χ-sliceness and lattice embeddings

Let Y be a $\mathbb{Q}S^3$ that bounds a $\mathbb{Q}B^4$, B.
(e.g. $Y = \Sigma_2(S^3, L)$, where L is a χ-slice link)

Also assume Y bounds a positive/negative definite 4-mfld X.

Then $Z = X \cup (-B)$ is a closed and positive/negative definite

By Donaldson's Diagonalization theorem Q_Z is diagonalizable. That is, $\exists$ a basis under which the matrix representation for Q_Z is $\pm I$.

Equivalently, $\exists$ a lattice isomorphism

$(H_2(Z), Q_Z) \cong (\mathbb{Z}^n, \pm I)$, where $n = \text{rank}(H_2(Z)) = \text{rank}(H_2(X))$

Since $X \subset Z$, there is a homomorphism $H_2(X) \to H_2(Z)$ induced by inclusion.
It follows that $\exists$ a lattice embedding

$$\varphi : (H_2(X), Q_X) \longrightarrow (\mathbb{Z}^n, \pm I)$$

Donaldson's Obstruction:

If Y is a $\mathbb{Q}S^3$ that bounds a positive/negative definite 4-manifold X and $\nexists$ lattice embedding $(H_2(X), Q_X) \to (\mathbb{Z}^n, \pm I)$, where $n = \operatorname{rank}(H_2(X))$, then Y does not bound a $\mathbb{Q}B^4$!

Moreover, if $Y = \Sigma_2(S^3, L)$ for some link $L \subset S^3$, then L is not χ-slice.

Determinant Obstruction (weaker but easier than Donaldson's)

Recall that if $\exists$ lattice embedding $(H_2(X), Q_X) \to (\mathbb{Z}^n, \pm I)$, then $|\det Q_X|$ is a square.

So, if $|\det Q_X|$ is not a square, then Y does not bound a $\mathbb{Q}B^4$.

Moreover, if $Y = \Sigma_2(S^3, L)$, then L is not χ-slice.

Recall: If $|\det L|$ is not a square, then L is not χ-slice.

A connection: If $L \subset S^3$ with $\det L \neq 0$ and X is a 4-manifold with $\partial X = \Sigma_2(S^3, L)$, then

$$|\det L| = |H_1(\Sigma_2(S^3, L))| = |\det Q_X|$$

Now suppose $Y = \Sigma_2(S^3, L)$ for some alternating link L with $\det L \neq 0$

Let F be a checkerboard surface for L.

If L is χ-slice, then the lattice embedding given above is cubiquitous.

(there is a generalization of this to cases in which L is not alternating)

Cubiquity Obstruction (Greene-Owens):

Let L be an alternating link and let F be a checkerboard surface for L.

If $\nexists$ a cubiquitous lattice embedding

$$(H_2(\Sigma_2(B^4, F)), Q_{\Sigma_2(B^4, F)}) \longrightarrow (\mathbb{Z}^n, \pm I), \quad n = \operatorname{rank}(H_2(\Sigma_2(B^4, F)))$$

then $\Sigma_2(S^3, L)$ does not bound a $\mathbb{Q}B^4$

(and L is not χ-slice)

Conjecture (Greene-Owens). The converse is true.

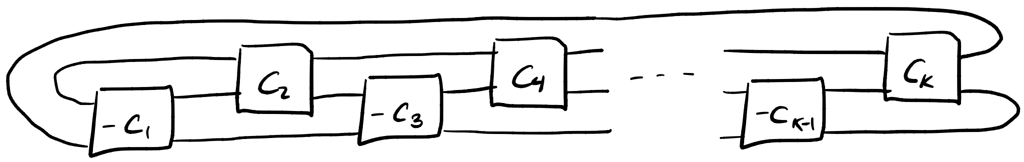
If $\exists$ a cubiquitous lattice embedding

$$(H_2(\Sigma_2(B^4, F)), Q_{\Sigma_2(B^4, F)}) \to (\mathbb{Z}^n, \pm I), \quad n = \operatorname{rank}(H_2(\Sigma_2(B^4, F)))$$

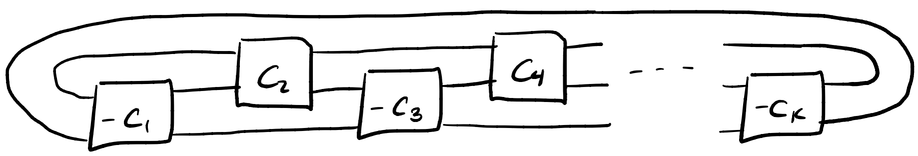
then $\Sigma_2(S^3, L)$ bounds a $\mathbb{Q}B^4$.

Lens Spaces and 2-bridge links.

A 2-bridge link $L(c_1, \dots, c_k)$ is an alternating link of the form



if k is even



if k is odd

where $c_i \geq 2 \;\; \forall i.$

Let $$\frac{p}{q} = [c_1, \dots, c_k]^+ = c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \dots \frac{1}{c_k}}}$$

Fact: $\Sigma_2(S^3, L(c_1, \dots, c_n)) = L(p,q)$ (lens space)

Let $$\frac{p}{q} = [a_1, \dots, a_n]^- = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \dots \frac{1}{a_n}}}$$

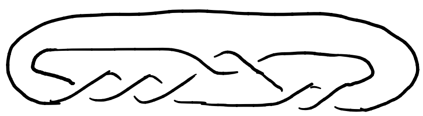
and $\frac{p}{p-q} = [b_1, \dots, b_m]^-$.

Let B and W be the checkerboard surfaces for $L(c_1,\dots,c_n)$

Then $Q_{\Sigma_2(B^4,W)}$ is represented by G_W (given by $\overset{-a_1}{\bullet}\overset{+1}{\text{———}}\overset{-a_2}{\bullet} - - - \overset{-a_n}{\bullet}$)

and $Q_{\Sigma_2(B^4,B)}$ is represented by G_B (given by $\overset{b_1}{\bullet}\overset{-1}{\text{———}}\overset{b_2}{\bullet} - - - \overset{b_m}{\bullet}$)

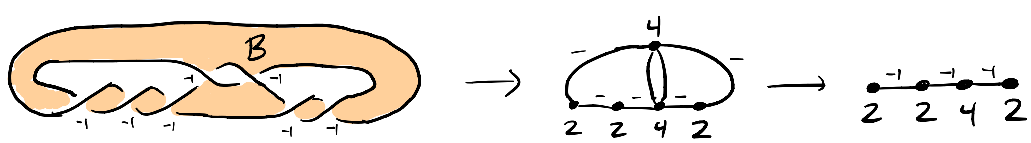
Moreover, $Q_{\Sigma_2(B^4,W)}$ is negative definite and $Q_{\Sigma_2(B^4,B)}$ is positive definite.

Ex: $L(3,2,2) =$ 

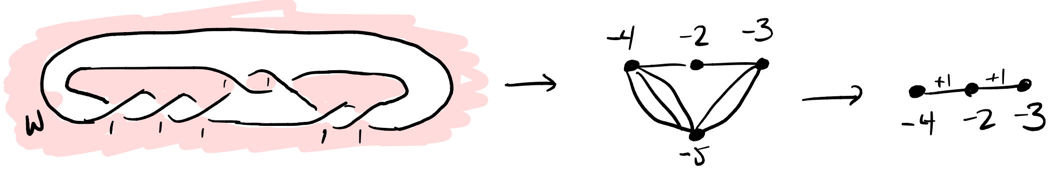
$$[3,2,2]^+ = 3 + \cfrac{1}{2+\cfrac{1}{2}} = \frac{17}{5} = \frac{p}{q} \Rightarrow \Sigma_2(S^3, L(3,2,2)) = L(17,5)$$

$$\frac{17}{5} = 4 - \cfrac{1}{2-\cfrac{1}{3}} = [4,2,3]^-$$

$$\frac{17}{17-5} = \frac{17}{12} = 2 - \cfrac{1}{2-\cfrac{1}{4-\cfrac{1}{2}}} = [2,2,4,2]^-$$



$\Rightarrow$ Intersection form of $\Sigma_2(B^4,B)$ is represented by the incidence matrix of $\overset{2}{\bullet}\overset{-1}{\text{—}}\overset{2}{\bullet}\overset{-1}{\text{—}}\overset{4}{\bullet}\overset{-1}{\text{—}}\overset{2}{\bullet}$ (i.e. G_W)



$\Rightarrow$ Intersection form of $\Sigma_2(B^4,W)$ is represented by the incidence matrix of $\underset{-4}{\bullet}\text{———}\underset{-2}{\bullet}\text{———}\underset{-3}{\bullet}$ (i.e. G_B)

Lisca classified which lens spaces bound $\mathbb{Q}B^4$'s

proof: • Obstruct all but 8 infinite families of lens spaces from bounding $\mathbb{Q}B^4$'s by showing $\nexists$ both lattice embeddings

$$(H_2(\Sigma_2(B^4, W)), G_W) \longrightarrow (\mathbb{Z}^{\operatorname{rank}(H_2(\Sigma_2(B^4, W)))}, -I)$$

$$(H_2(\Sigma_2(B^4, B)), G_B) \longrightarrow (\mathbb{Z}^{\operatorname{rank}(H_2(\Sigma_2(B^4, B)))}, I).$$

(The embeddings that do exist give rise to standard subsets)

• Show that the remaining 8 infinite families bound $\mathbb{Q}B^4$'s by showing that the associated 2-bridge links are χ-slice. $\blacksquare$

As a consequence,

$$L(p,q) \text{ bounds a } \mathbb{Q}B^4 \iff L(c_1, \dots, c_n) \text{ is } \chi\text{-slice}$$

Project: Is Greene-Owens' Conjecture true for 2-bridge links?

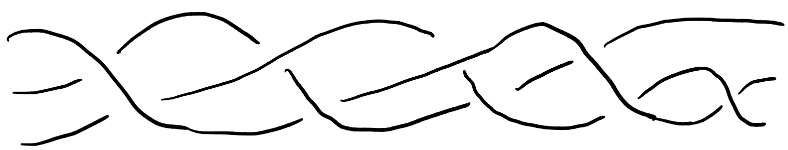
It is known to be true when $\sum(a_i - 3) < 0$.

What about when $\sum(a_i - 3) \geq 0$?

Phrased differently: which standard subsets with $I(S) \geq 0$ generate cubiquitous lattice embeddings?

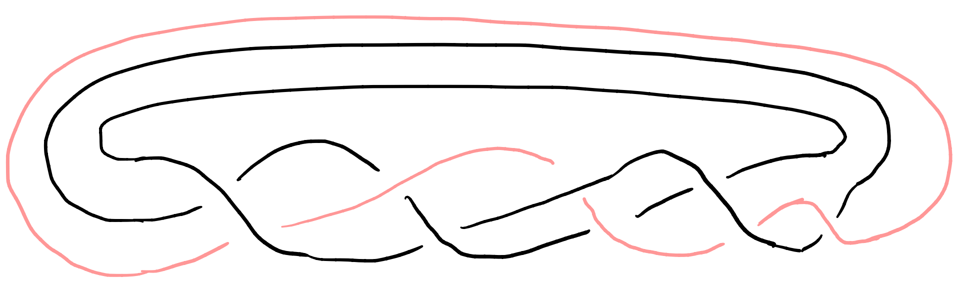
Alternating braid closures

A braid is obtained by taking n parallel strands and adding crossings

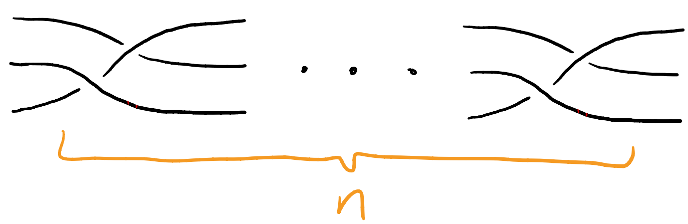


(3-braid)

The closure of a braid is a link:



Recall, a wheel link L_n is the closure of the 3-braid



If n is even, L_n is not χ-slice.

One can check that $\det L_n$ is not a square or that $\nexists$ an associated lattice embedding.

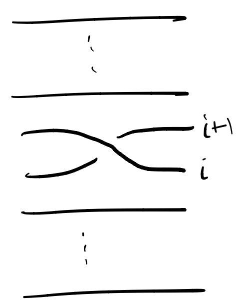
In particular $\Sigma_2(S^3, L_n)$ doesn't bound a $\mathbb{Q}B^4$!

On HW you showed that L_3 and L_5 are χ-slice and so $\Sigma_2(S^3, L_i)$ bounds $\mathbb{Q}B^4$ for $i = 3, 5$.

It turns out that $\Sigma(S^3, L_n)$ bounds $\mathbb{Q}B^4$ $\forall$ odd n, but not all L_n are χ-slice (for n odd)

Moreover, the Greene–Owens conjecture is true for odd wheel links

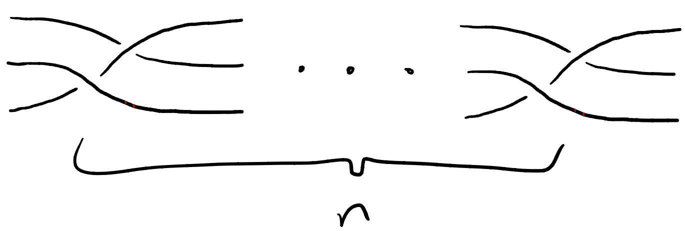
Notation: Let $\sigma_i =$



for $1 \le i \le n-1$

Then any braid can be expressed as a word in $\sigma_1, \dots, \sigma_{n-1}, \sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}$

Ex:



$= (\sigma_1 \sigma_2^{-1})^n$

Project: Let K be odd. Let $L_{K,n}$ be the closure of the alternating K-braid given by $(\sigma_1 \sigma_2^{-1} \cdots \sigma_{K-2} \sigma_{K-1}^{-1})^n$

Explore the χ-sliceness of $L_{K,n}$ and the Greene–Owens conjecture.