Cubiquitous Lattices and χ-sliceness

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Background

It is a classical result that any knot or link in $\mathbb{R}^3$ bounds a surface embedded in $\mathbb{R}^3$. It turns out that knots and links also bound surfaces embeddeded in $\mathbb{R}^4$. An active area of research explores knots and links that bound surfaces with simple topology.

Definition (Slice). A knot $K \subset S^3$ is *slice* if K bounds a smoothly properly embedded disk $D \subset B^4$.

Figure 1 shows a knot that bounds a disk in $\mathbb{R}^3$; however, the disk is not embedded since it intersects itself. After pushing the disk into $\mathbb{R}^4$, it no longer intersects itself. Thus the knot is slice.

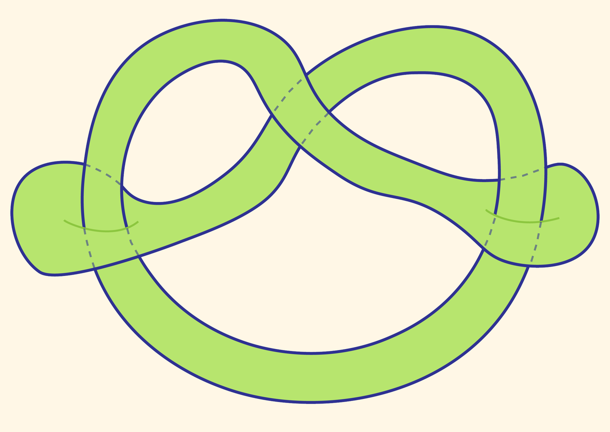

Fig. 1: Example surface bounded by a slice knot

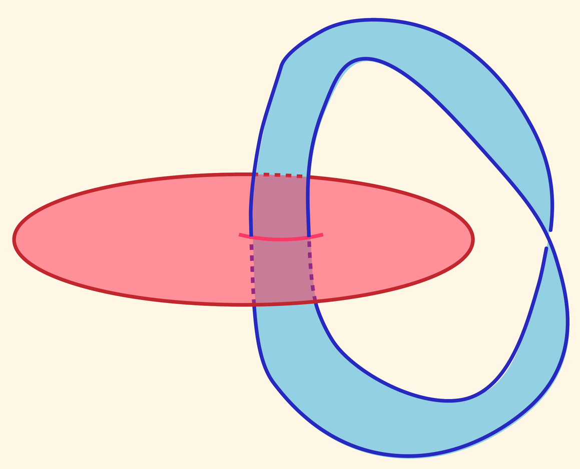

Fig. 2: Example χ-slice link

The notion of χ-sliceness is a generalization of sliceness for links.

Definition (χ-slice). A link $L \subseteq S^3$ is *χ-slice* if L bounds a smoothly properly embedded surface $F \subseteq B^4$ with Euler characteristic 1 and no closed components.

In order to understand which links are χ-slice, one must apply algebraic tools (called *obstructions*) that can inform us that particular links are not χ-slice. It turns out that integer sublattices of $\mathbb{Z}^n$ serve as an obstruction.

Definition (Cubiquitous). A sublattice $\Lambda \subset \mathbb{Z}^n$ is called *cubiquitous* if every unit cube in $\mathbb{Z}^n$ has a nonempty intersection with Λ [2].

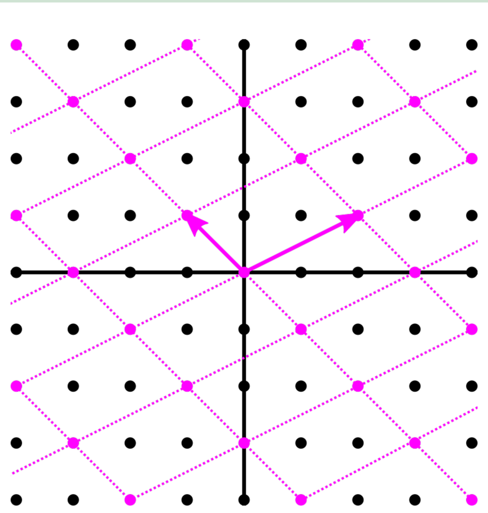

Fig. 3: Cubiquitous sublattice

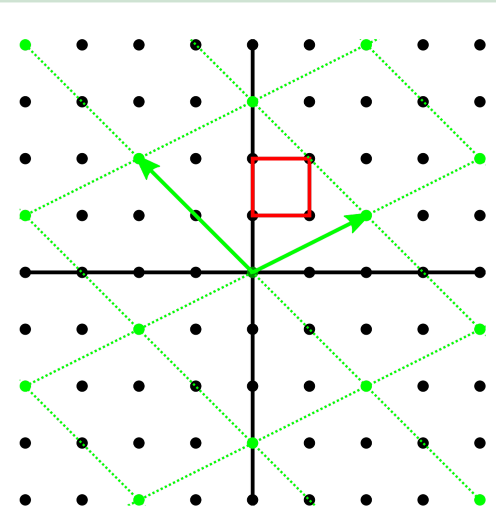

Fig. 4: Noncubiquitous sublattice

Theorem (Greene-Jabuka [1], Greene-Owens [2]). If L is an alternating nonsplit χ-slice link, then there exists an associated cubiquitous sublattice $\Lambda(L)$ of $\mathbb{Z}^n$.

Definition (Wu Element). For a sublattice $\Lambda \subset \mathbb{Z}^n$ with a basis $B = \{v_1, ..., v_n\}$ we define the *Wu element* as

$$W = \sum_{i=1}^{n} v_i$$

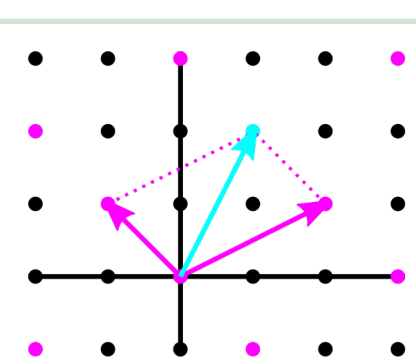

Fig. 5: Wu element of $(2,1)$ and $(-1,1)$

Motivation

One can associate a 3-manifold $\Sigma(L)$ to any link L. If L is an alternating nonsplit $\chi-$slice link, then $\Sigma(L)$ is the boundary of a simple 4-manifold called a *rational 4-ball*.

Theorem ([1], [2]). Given an alternating nonsplit link L, if $\Sigma(L)$ bounds a rational 4-ball, then there exists an associated cubiquitous sublattice $\Lambda(L)$ of $\mathbb{Z}^n$.

Greene-Owens Conjecture. Given an alternating nonsplit link L, if there exists an associated cubiqitous sublattice $\Lambda(L)$ of $\mathbb{Z}^n$, then $\Sigma(L)$ bounds a rational 4-ball.

Questions:

What conditions ensure or obstruct cubiquity?

Is the Greene-Owens Conjecture true for torus links?

Cubiquity Results

Lemma 1. Let $\Lambda \subset \mathbb{Z}^n$ be a cubiquitous lattice. Then there exists a basis B for Λ such that each $b \in B$ satisfies

$$\|b\|^2 = b_1^2 + ... + b_n^2 \leq n + 3$$

Lemma 2. Let $\Lambda \subset \mathbb{Z}^n$ be a cubiquitous lattice. Then there exists a basis B with entries in $\{-2, -1, 0, 1, 2\}$ such that each $b \in B$ has at most one entry with absolute value 2.

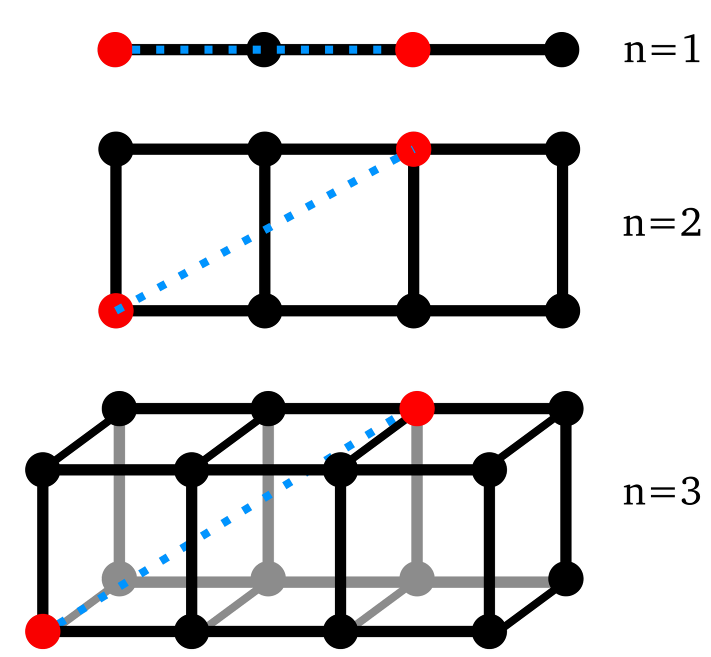

Fig. 6: Intuition for Lemma 1

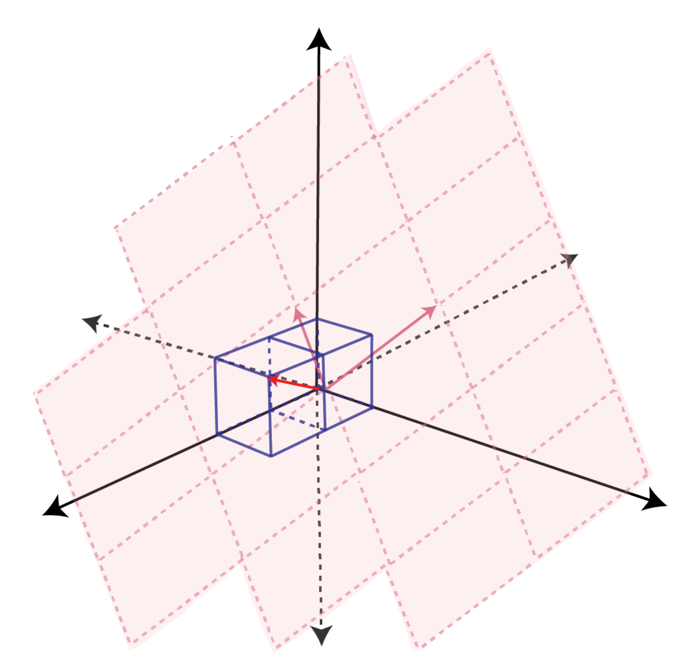

Fig. 7: 3D Intuition for Lemma 2

Proposition. Let Λ be an orthogonal sublattice with orthogonal basis B. Denote the Wu element by $W = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ and let O be the number of odd entries of W. If $\sum_{i=1}^{n} z_i^2 > 4n - 3O$, then Λ is not cubiquitous.

Definition. A sublattice $\Lambda \subset \mathbb{Z}^n$ is called *orthogonal* if it admits an orthogonal basis.

Theorem. Let Λ be orthogonal. Then Λ is cubiquitous if and only if it admits an orthogonal basis $\mathcal{B} = \{v_1, ... v_n\}$ such that the matrix

$$B = \begin{bmatrix} | & & | \\ v_1 & \ldots & v_n \\ | & & | \end{bmatrix}$$

is a block diagonal matrix and has blocks of the form $[1]$, $[2]$, or $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ up to reordering and negating the standard orthonormal basis vectors of $\mathbb{Z}^n$.

Applications

Our theorem has the following implication on connect sums of torus links.

Corollary. The Greene-Owens Conjecture is true for connected sums of alternating positive torus links. Moreover, a connected sum of alternating positive torus links is χ-slice if and only if the summands are of the form $T(2,1), T(2,4)$, and $T(2,2)\#T(2,2)$.

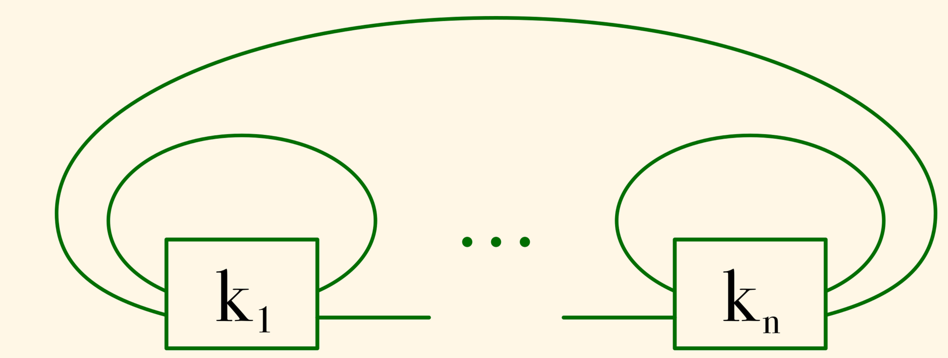

Fig. 8: An example of a connected sum of torus links. Each k_i in the box indicates the number of half twists.

Methods

To understand cubiquity in higher dimensions, we utilized the result from Lemma 2 to make a computationally reasonable set of orthogonal bases to check against the cubiquity obstruction given in the Proposition. This mainly entailed two algorithms: an orthongal basis generator and a Wu obstruction checker. The code and documentation are available at https://github.com/ericaychoi/cubiquity-check.

Next Steps

Currently, we are exploring the cubiquity of sublattices with *mediocre* bases, which include othogonal bases.

Definition (Mediocre Subset). A set $S = \{v_1, ..., v_n\}$ of linearly independent vectors in $\mathbb{Z}^n$ is *mediocre* if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & |i-j| \geq 1 \\ 0, -1 & |i-j| = 1 \end{cases}$$

A *mediocre sublattice* of $\mathbb{Z}^n$ is a sublattice a mediocre basis.

These subsets share many properties with *good subsets* that were explored by Lisca [3]. We hope to generalize our results on orthogonal lattices to mediocre lattices.

Acknowledgements

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References

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