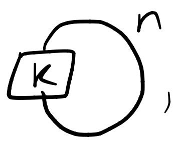
Rational Homology 4-balls

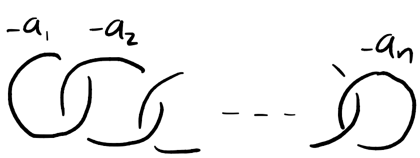
Def: A manifold X is called a rational homology Y if $\operatorname{rank}(H_i(X)) = \operatorname{rank}(H_i(Y))$ $\forall i$.

That is, $H_i(X;\mathbb{Q}) \cong H_i(Y;\mathbb{Q})$ $\forall i$.

Rational homology 3-spheres ($\mathbb{Q}S^3$'s) is a large family of 3-manifolds with simple topology (as measured by rational homology).

Ex: Let $X =$ 

, $n \neq 0$. Then ∂X is a $\mathbb{Q}S^3$

Ex: Let $X =$ 

, $a_i \geq 2$ $\forall i$.

Then ∂X is a lens space $L(p,q)$, where

$$\frac{p}{q} = [a_1, \dots, a_n]^- = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 \cdots - \cfrac{1}{a_n}}}$$

The simplest 4-manifold (as measured by rational homology) that a $\mathbb{Q}S^3$ could bound is a rational homology 4-ball ($\mathbb{Q}B^4$).

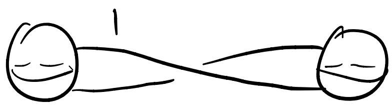
Fact: If X is a $\mathbb{Q}B^4$, then ∂X is a $\mathbb{Q}S^3$.

The converse is not true in general. The following question is an active area of research

Question: Which $\mathbb{Q}S^3$s bound $\mathbb{Q}B^4$s?

This is solved for certain families of $\mathbb{Q}S^3$s (e.g. lens spaces), but not in general.

Ex: • B^4 is a $\mathbb{Q}B^4$

•  is a $\mathbb{Q}B^4$ whose boundary is the lens space $L(4,1)$

Fact: Let $L \subset S^3$ be a link.

$\Sigma_2(S^3, L)$ is a $\mathbb{Q}S^3 \iff \det L \neq 0$

Theorem: If L is χ-slice and $\det L \neq 0$, then $\Sigma_2(S^3, L)$ bounds a $\mathbb{Q}B^4$, namely $\Sigma_2(B^4, F)$, where F is a χ-slice surface for L.

Ex: $\Sigma_2(S^3, 6_1)$ bounds a $\mathbb{Q}B^4$ since 6_1 is slice.