

Math 425 Review Session Final

#1) Compute line integral C is unit circle, counterclockwise

$$\int_C \vec{F} \cdot d\vec{s} \quad F(x, y) = (y, -x)$$

1. Use Definition

Param C

$$\vec{c}(t) = (\cos t, \sin t), \quad 0 \leq t \leq 2\pi$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{s} &= \int_C \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t) dt = \int_0^{2\pi} (\sin t, -\cos t) \cdot (-\sin t, \cos t) dt \\ &= \int_0^{2\pi} -1 dt = \boxed{-2\pi} \end{aligned}$$

2. Fund Thm of Line Integral

Need \vec{F} to be conservative

$$\int_C \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

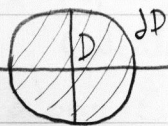
$$Q: \vec{F}(x, y) = \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \text{ conservative?}$$

$$\text{Check whether } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\text{Since } \frac{\partial P}{\partial y} = 1 \neq \frac{\partial Q}{\partial x} = -1, \vec{F} \text{ not conservative}$$

For vector fields in \mathbb{R}^3 , use curl \vec{F}

3. Green's Theorem $D \subset \mathbb{R}^2$, $\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$



Another way to write $\int_{\partial D} P dx + Q dy$

$$\begin{aligned} \iint_C \vec{F} \cdot d\vec{s} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (-1 - 1) dA = \iint_D -2 dA \\ &= \int_0^{2\pi} \int_0^1 -2r dr d\theta = \boxed{-2\pi} \end{aligned}$$

! Make sure to check orientation !

#2) Let $\vec{F}(x, y, z) = (2x, x^2, z)$

Calculate the Flux of \vec{F} through the boundary of the solid bounded by $x^2 + y^2 = 1$, $z = 0$, and $z = 2 - x^2 - y^2$ (oriented outward)

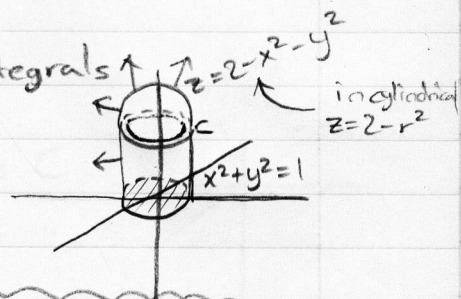
We need to calculate $\iint_S \vec{F} \cdot d\vec{s}$

To calculate this directly, need 3 integrals

Instead, use Divergence Theorem:

$$\begin{aligned} \iint_{\partial W} \vec{F} \cdot d\vec{s} &= \iiint_W \operatorname{div} \vec{F} \, dV = \iiint_W 3 \, dV \\ &= \int_0^{2\pi} \int_0^1 \int_0^{2-r^2} 3r \, dz \, dr \, d\theta \end{aligned}$$

change to cylindrical, and remember extra r



! Double check orientation !

$$= \boxed{\frac{9\pi}{2}}$$

To apply Div Thm: ∂W oriented away from W (see arrows on graphs)

$$z = 2 - 1 = 1$$

$$z = 2 - (x^2 + y^2)$$

#3) Let C be a curve of intersection of $z = 2 - x^2 - y^2$ & $x^2 + y^2 = 1$.

Calculate the work done by \vec{F} in moving a particle along C . (line integral)

Calculate $\int_C \vec{F} \cdot d\vec{s}$

1. Use Definition

Param C : See blue parts, next

$$\vec{r}(t) = (\cos t, \sin t, 1)$$

! Double Check Orientation !

In this case, wrong orientation, therefore negate at end and label in answer.

$$\int_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (2\cos t, \cos^2 t, 1) \cdot (-\sin t, \cos t, 0) dt$$

$$= \int_0^{2\pi} -2\cos t \sin t + \cos^2 t \, dt \quad \text{Too hard, instead use Stokes}$$

2. Stokes

$$\int_S \vec{F} \cdot d\vec{s} = \iint_S \text{curl } \vec{F} \cdot d\vec{s}$$

Find a simple surface whose boundary is C
 Fill in curve C , turns into a disk/surface

Let S be the surface $z=1$, $x^2+y^2 \leq 1$

Parametrize it: $\vec{r}(u,v) = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$, $u^2+v^2 \leq 1$

$$\vec{r}_u = (1, 0, 0)$$

$$\vec{r}_v = (0, 1, 0)$$

$$\vec{r}_u \times \vec{r}_v = (0, 0, 1)$$

! Orientation Check !

Orientation is upward pointing, therefore wrong because surface is on right as walking along C , negate final answer

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & x^2 & z \end{vmatrix} = (0, 0, 2x)$$

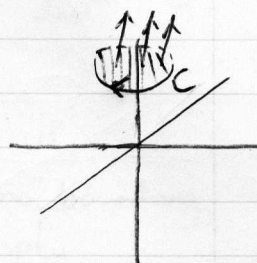
Need to compute:

$$\iint_S \text{curl } \vec{F} \cdot d\vec{s} = \iint_D (0, 0, 2u) \cdot (0, 0, 1) \, dA = \iint_D 2u \, du \, dv$$

Switch to polar: $u = r \cos \theta$
 $v = r \sin \theta$

$$= \int_0^{2\pi} \int_0^1 2r \cos(\theta) r \, dr \, d\theta$$

remember
to negate
final answer



Green Stokes

$$\int_{\partial M} w = \int_M dw \iff (k-1)\text{-form } w \text{ on } M (k\text{-dimensional})$$

means there is no boundary.

If α is an exact k -form, M is closed, show $\int_M \alpha = 0$

What does this mean?

α exact $\Rightarrow \alpha = dw$ for some $(k-1)$ -form w

Integrate over boundary \rightarrow

$$\int_M \alpha = \int_M dw = \int_{\partial M} w = 0 \quad \text{since } M \text{ has no boundary so } \partial M \text{ is empty}$$

0-form = function

1-form on \mathbb{R}^n $\alpha = g_1 dx_1 + \dots + g_n dx_n$

Ex: on \mathbb{R}^3 , $\alpha = x^2 y dx + \sin z dy + 2x dz$

2-form on \mathbb{R}^3 : $\beta = g_1 dx dy + g_2 dx dz + g_3 dy dz$

on \mathbb{R}^4 : $\beta = g_1 dx dy + g_2 dx dz + g_3 dx dw + g_4 dy dz + g_5 dy dw + g_6 dz dw$

3-form on \mathbb{R}^3 : $w = f dx dy dz$

on \mathbb{R}^4 : $\omega = f_1 dx dy dz + f_2 dx dy dw + f_3 dx dz dw + f_4 dy dz dw$