Ribbon Surfaces and Sliceness

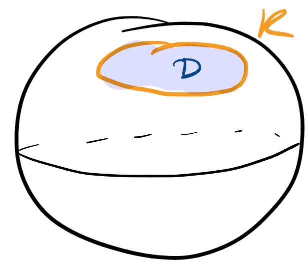
Recall: Any knot/link L bounds an embedded orientable (and nonorientable) surface F in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

no self intersections

Notation: $\partial F = L$

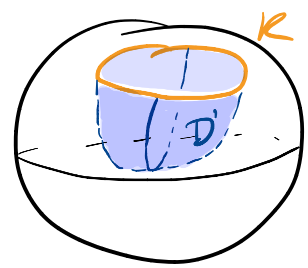
Fact: The only knot that bounds a disk in S^3 is the unknot

Motivating Example:

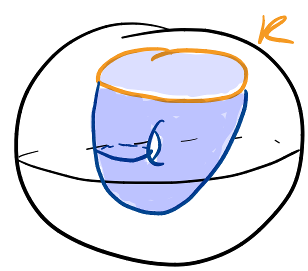


Let K be a knot in $S^2 = \mathbb{R}^2 \cup \{\infty\}$
(it must be the unknot)

Let D be a disk in S^2 bounded by K



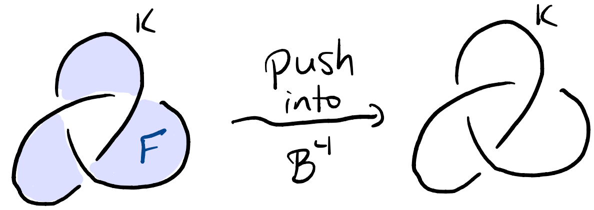
Then D can be pushed into B^3 giving a disk D' in B^3 bounded by K, which is still in S^2



Moreover, K can bound other kinds of surfaces in B^3 that it cannot bound in S^2

The same is true for knots and links in S^3:

Any surface F bounded by a link L in S^3 can be pushed in B^4.



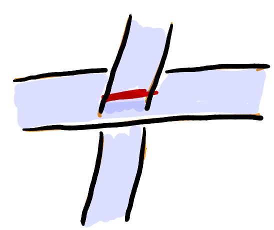
Moreover, L can potentially bound more kinds of surfaces in B^4 than in S^3.

Simply pushing embedded surfaces into B^4 is not very interesting, since you don't need B^4 to see them.

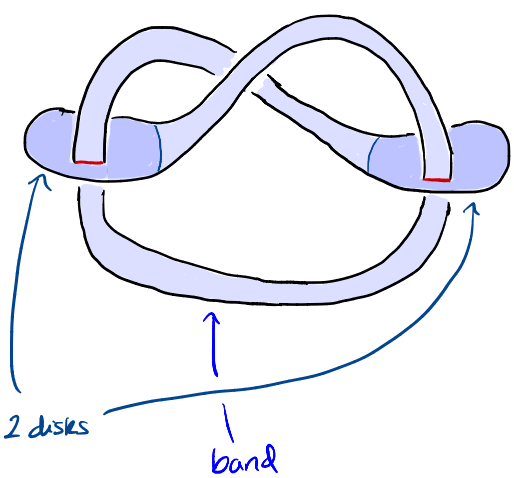
Here is a more interesting class of surfaces:

Def: A ribbon surface in S^3 is an immersed surface that locally is either embedded in S^3 or intersects itself in a ribbon singularity

might have self-intersections

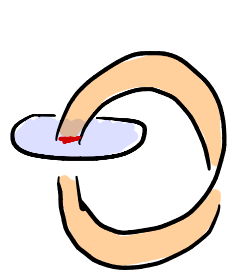


Ex:



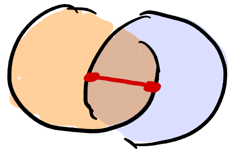
orientable, $\chi = 2-1=1$, one boundary component

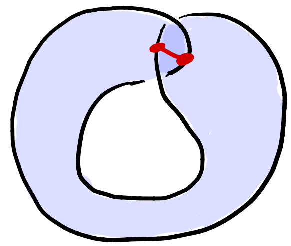
$\Rightarrow$ ribbon disk



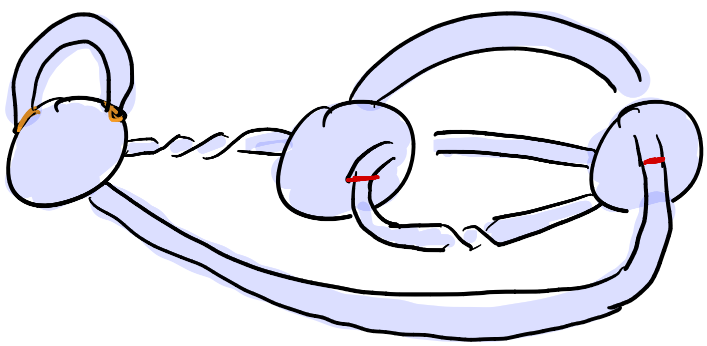
Disk $\sqcup$ Mobius band

Ex: Immersed surfaces that are not ribbon



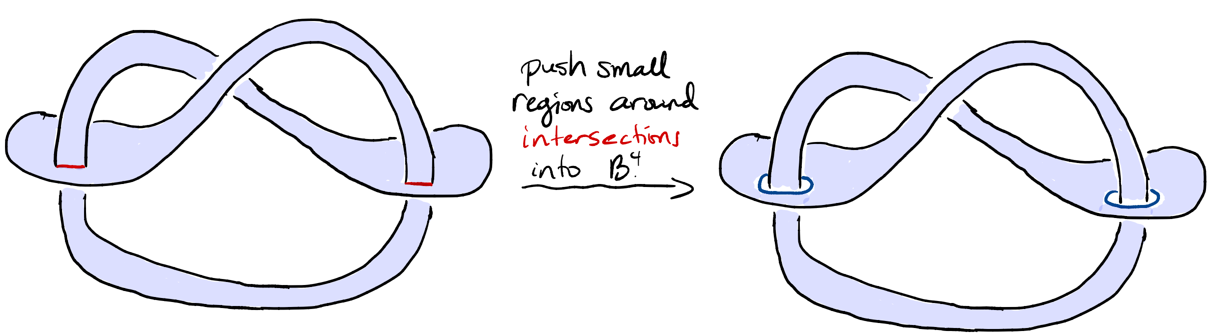


To build a ribbon surface,
Start with n disks and attach bands



$\chi = 3 - 4 = -1$

Key: A ribbon surface can be pushed into B^4 to remove the self-intersections (i.e. make it embedded)



So ribbon surfaces are embedded surfaces in B^4
(but not in S^3).

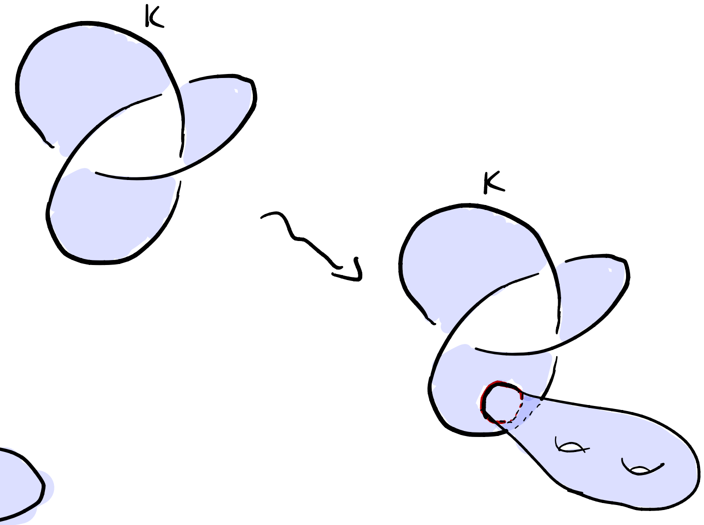
Def: A knot is called slice if it bounds a smooth properly embedded disk in B^4.

Why a disk?

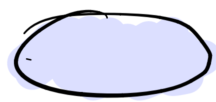
The simplest surface a knot can bound is a disk.

It is hard to bound simple surfaces

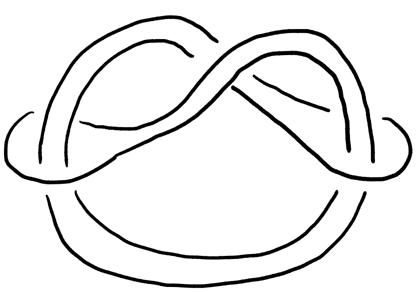
It is easy to bound more complicated surfaces by performing connected sums



Ex: The unknot is slice since it bounds a disk in S^3 that we can push into B^4



Ex:

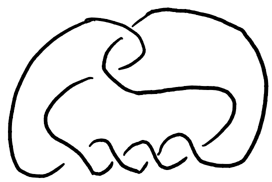


is slice. It bounds the ribbon disk seen earlier.

Def: A knot is called ribbon if it bounds a ribbon disk.

By definition, ribbon $\Rightarrow$ slice.

Slice-Ribbon Conjecture: slice $\Rightarrow$ ribbon.

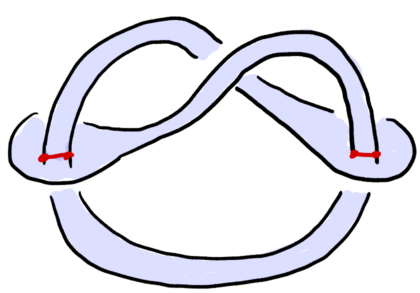
Ex:  is slice. How can we tell?

We can show this knot is ribbon (and hence slice) by constructing a ribbon disk.

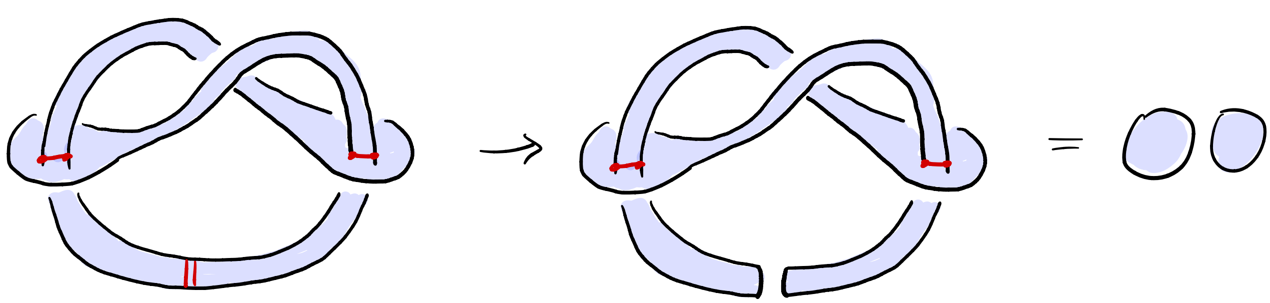
Band Moves

Idea: Deconstruct the ribbon disk.

To construct a ribbon disk, start with n disks and attach $n-1$ bands (so $\chi=1$)

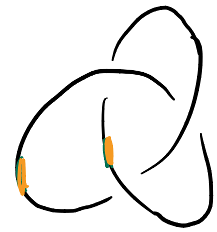


To deconstruct, cut through the bands to recover the disks

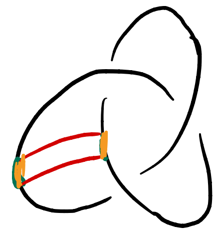


More generally, given a knot K, a band move is the following process.

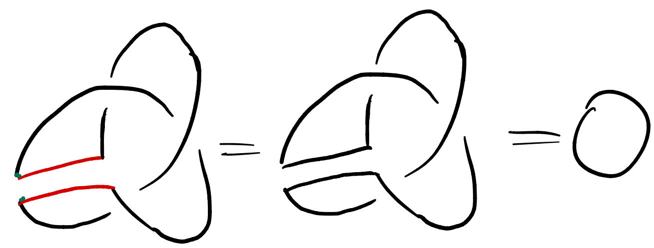
1.) pick two arcs on K



2.) draw two arcs between the endpoints of the two arcs on K.

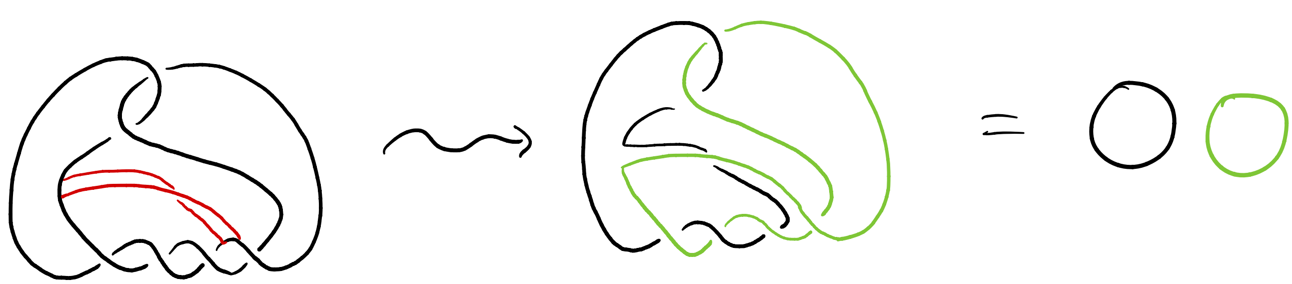


3.) Erase the arcs on K

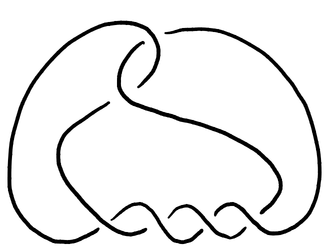


Fact: if we perform n band moves and get the $n+1$ component unlink, then K is ribbon (and hence slice)

Ex:



$\Longrightarrow$



is slice!

Two versions of sliceness for links

<u>Def</u>: An n component link is called <u>slice</u> (<u>ribbon</u>) if it bounds the disjoint union of n smoothly embedded (ribbon) disks in B^4.

<u>Ex</u>: the unlink is slice

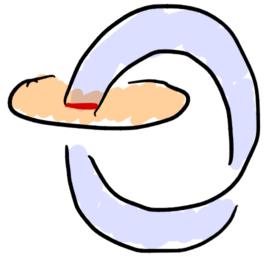
<u>Def</u>: A link $L \subset S^3$ is called <u>χ-slice</u> if it bounds a smooth properly embedded surface $F \subset B^4$ with no closed components and $\chi(F) = 1$
Such an F is called a <u>χ-slice surface for F</u>.
If F is ribbon, L is called <u>χ-ribbon</u>.

If L is a knot, then χ-slice $=$ slice
since the only surface with one boundary component and $\chi = 1$ is the disk.

So both notions of sliceness of links generalize the notion of slice knots.

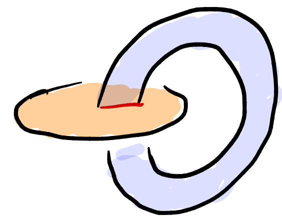
We will focus on χ-slice links because of their utility in constructing particular 4-manifolds with simple topology called rational homology 4-balls, which we will discuss later.

Ex:



$F = \text{Disk} \sqcup \text{Mobius band}$

$\chi(F) = 1 + 0 = 1$



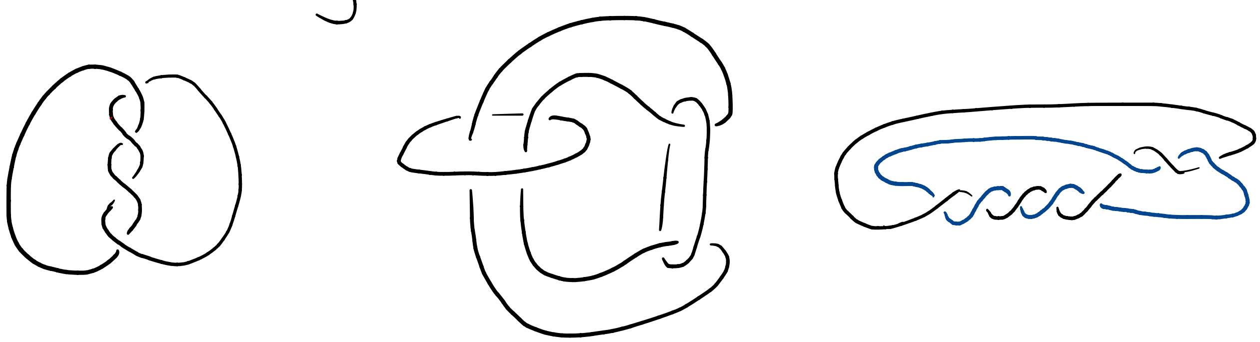
$F = \text{Disk} \sqcup \text{Annulus}$

$\chi(F) = 1$

As with slice knots, one way to describe a ribbon surface for L with $\chi = 1$ is by performing band moves. In particular:

If performing n band moves on L yield the $n+1$ component unlink, then L is χ-slice.

Ex: The following are χ-slice



Band Moves: