Let $KCS^3$ be an oriented knot.

A Seifert surface for $K$ is an oriented surface $FCS^3 \cup DF = K$.

**Theorem:** Every knot has a Seifert surface.

**Proof idea:**
1. Perform oriented resolutions at each crossing to get an unlink.
2. Add handles (handles) to disks according to original crossings.

$\chi = -1$ implies genus $1$.

**Proof:**

Given an oriented knot and a Seifert surface $F$ for $K$, pick a collection of simple closed curves $b_1, \ldots, b_g$ on $F$ that forms a basis for $H_1(F; \mathbb{Z}) \cong \mathbb{Z}^g$ (where $g =$ genus of $F$).

Let $V$ be the $2g \times 2g$ matrix whose $(ij)$-th entry is $\ell_K(a_i, \tilde{a}_j)$, where $\tilde{a}_j$ is a pushoff of $a_j$ off $F$ in the positive direction.

**Def:** The determinant of $K$ is $\det(K) := \det(V + V^T)$.

**Def:** The signature of $K$ is

$$\sigma(K) := \sigma(V + V^T) = \# \text{pos values} - \# \text{neg values}$$

**Note:** All eigenvalues of a symmetric matrix are real $\#s$.

This is why we compute $\sigma(V + V^T)$ and not $\sigma(V)$.
\[ \text{Ex:} \quad \text{genus}(F) = 1 \Rightarrow H_2(F) \cong \mathbb{Z}^2 \Rightarrow \text{Need 2 curves} \]

\[ \begin{align*}
\text{LK}(a, b) &= 1 \\
\text{LK}(a, b) &= 0 \\
\text{LK}(a, b) &= 1 \\
\text{LK}(b, b) &= 1
\end{align*} \]

\[ V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow V + V^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]

\[ \sigma(V + V^T) = 2 \]

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\[ \text{evals:} \]
\[ (2 - \lambda)^2 - 1 = 0 \]
\[ \lambda^2 - 4\lambda + 3 = 0 \]
\[ (\lambda - 3)(\lambda - 1) = 0 \]
\[ \lambda = 1, 3 \]

Properties:
- \( \sigma(K) \) doesn't depend on \( F \) or orientation on \( K \)
- \( \sigma(K) = -\sigma(m(K)) \) (where \( m(K) = \text{mirror of } K \))

**Thm:** If \( K \) is slice, then \( \sigma(K) = 0 \).

**Ex:** If \( p \geq 0 \) or \( \frac{1}{p} + \frac{1}{2} + \frac{1}{r} \leq 0 \), then \( \sigma(P(p, q, r)) \neq 0 \).
Given an oriented link $L$, we define $\sigma(L)$ in the same exact way. Note, however, that if we change the orientation of one of the link components, $\sigma(L)$ will change (so $\sigma(L)$ depends on orientation).

**Levine-Tristam Signatures**

Let $L \subset S^3$ be an oriented link, we $C$ with $\|w\|=1$ ($w \neq 1$). The $w$-signature of $L$ is

$$\sigma_w(L) = \sigma((1-w)V + (1-\bar{w})V^T)$$

where $V$ is a Seifert matrix for $L$.

**Note:**
- If $w = -1$ then $\sigma_{-1}(L) = \sigma(L)$.
- $(1-w)V + (1-\bar{w})V^T$ is a hermitian matrix (i.e., $V^T = V$). Such matrices have all real eigenvalues.

**Def:** The $w$-nullity of $L$ is

$$\eta_w(L) = \text{null}((1-w)V + (1-\bar{w})V^T) + (\text{# components of } F) - 1$$

$$(\text{dimension of null space})$$

where $F$ is a Seifert surface for $L$ and $V$ is its Seifert matrix.
Ex: Trefoil revisited: \( V = \begin{bmatrix} 1 & 1 \\ \end{bmatrix} \)

Let \( \omega \in \mathbb{C} \) be an \( n \)th root of unity.

Then
\[
(1-\omega) V + (1-\bar{\omega}) V^T = \begin{bmatrix} 2-(\omega+\bar{\omega}) & 1-\omega \\ 1-\bar{\omega} & 2-(\omega+\bar{\omega}) \end{bmatrix}
\]

Let \( \omega = a + bi \), \( \bar{\omega} = a - bi \). Note that \( a^2 + b^2 = 1 \) & \( a, b \leq 1 \)

\[
(1-\omega) V + (1-\bar{\omega}) V^T = \begin{bmatrix} 2-2a & 1-\omega \\ 1-\bar{\omega} & 2-2a \end{bmatrix}
\]

Eigenvalues:
\[
(2-2a-\lambda)^2 - (1-\omega)(1-\bar{\omega}) = 0
\]
\[
\lambda^2 - 2(2a-\lambda) + (2-2a)^2 - 1 + \omega + \bar{\omega} - 1 = 0
\]
\[
\lambda^2 - (4-4a)\lambda + 4a - 6a + 2 = 0
\]
\[
\lambda = \frac{4-4a \pm \sqrt{(4-4a)^2 - 4(4a^2-6a+2)}}{2}
\]
\[
= 2-2a \pm \sqrt{2-2a} > 0
\]

\( \Rightarrow \) \( \sigma_\omega(K) = 2 \) & \( \omega \)

\[
\begin{cases}
\text{If } a = -1 \text{ (so } b = 0 \text{ & } \omega = -1), & \lambda = 4 \pm 2 > 0 \text{ (same calculation as previous example)} \\
\end{cases}
\]

\( \Rightarrow \) \( \sigma_{-1}(K) = 2 = \sigma(K) \)

Nullity: \( \nu(K) = 0 + 1 - 1 = 0 \) & \( \omega \).
Thm: If det $L$ is not a perfect square, then $L$ is not $x$-slice.

Thm (Donald-Owen):
Let $L$ be an oriented link with $\mu_w(L)=0$ for some $w$. If $L$ bounds an oriented surface with $x=1$, then $\sigma_w(L)=0$.

Note: this only obstructs $L$ from bounding oriented surfaces with $x=1$. Often we can do more.

Ex: $L=P(2,2,2)$

If $L$ bounds a nonorientable surface $F$ with $x=1$, then $F=\text{Mobius} \cup \text{Mobius} \cup \text{Disk}$ (check)

If we remove one of the Mobius bands, then we have a new surface $F=\text{Mobius} \cup \text{Disk}$ bounded by Hopf link. But, we know Hopf link is not $x$-slice since det $L \neq 0$

$\Rightarrow L$ not $x$-slice