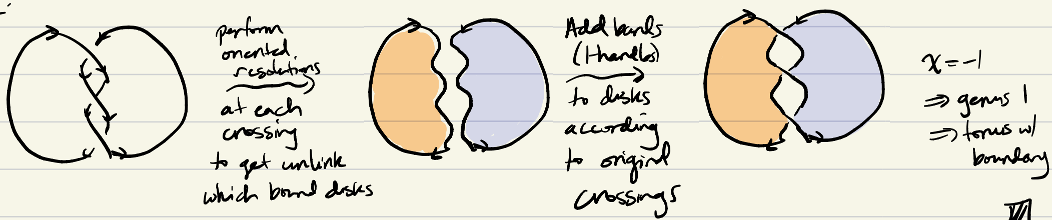
Signature

Let $K \subset S^3$ be an oriented knot

A <u>Seifert surface</u> for K is an oriented surface $F \subset S^3$ w/ $\partial F = K$

Thm: Every knot has a Seifert surface

proof idea:



∎

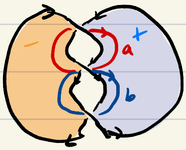
Given an oriented knot and a Seifert surface F for K, pick a collection of simple closed curves $\{b_1, \ldots, b_{2g}\}$ on F that forms a basis for $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ (where g = genus of F)

Let V be the $2g \times 2g$ matrix whose (i,j)-th entry is $lk(a_i, \tilde{a}_j)$, where $\tilde{a}_j$ is a pushoff of a_j off F in the positive-direction

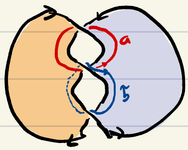
Def: The <u>determinant of K</u> is $\det(K) := \det(V + V^T)$

Def: The <u>signature of K</u> is
$$\sigma(K) := \sigma(V + V^T) = \#\text{pos evalues} - \#\text{neg evalues}$$

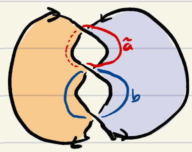
Note: All evalues of a symmetric matrix are real #s.
This is why we compute $\sigma(V+V^T)$ and not $\sigma(V)$.

Ex: 

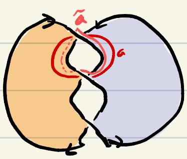
genus$(F)=1 \Rightarrow H_2(F)\cong \mathbb{Z}^2 \rightarrow$ Need 2 curves



$lk(a,\tilde{b})=1$



$lk(\tilde{a},b)=0$



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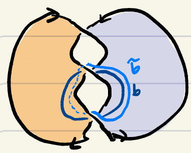


$lk(b,\tilde{b})=1$

$$V=\begin{bmatrix}1&1\\0&1\end{bmatrix} \Rightarrow V+V^T=\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

evalues:

$(2-\lambda)^2-1=0$

$\lambda^2-4\lambda+3=0$

$(\lambda-3)(\lambda-1)=0 \quad \lambda=1,3$

$$\sigma(V+V^T)=2$$

Properties:
- $\sigma(K)$ doesn't depend on F or orientation on K
- $\sigma(K)=-\sigma(m(K))$ (where $m(K)$ = mirror of K)

Thm: If K is slice, then $\sigma(K)=0$.

Ex: If $p,q,r>0$ or $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<0$, then $\sigma(P(p,q,r))\neq 0$.

Given an oriented link L, we define $\sigma(L)$ in the same exact way. Note, however, that if we change the orientation of one of the link components, $\sigma(L)$ will change (so $\sigma(L)$ depends on orientation)

Levine-Tristam Signatures

Let $L \subset S^3$ be an oriented link, $\omega \in \mathbb{C}$ with $\|\omega\| = 1$ $(\omega \neq 1)$
The ω-signature of L is

$$\sigma_\omega(L) = \sigma\left((1-\omega)V + (1-\bar{\omega})V^T\right)$$

where V is a Seifert matrix for L.

Note:
- If $\omega = -1$ then $\sigma_{-1}(L) = \sigma(L)$.
- $(1-\omega)V + (1-\bar{\omega})V^T$ is a hermitian matrix (ie. $\bar{V}^T = V$)
  Such matrices have all real eigenvalues

Def: The ω-nullity of L is

$$\eta_\omega(L) = \text{null}\left((1-\omega)V + (1-\bar{\omega})V^T\right) + (\#\text{components of } F) - 1$$

(dimension of null space)

where F is a Seifert surface for L and V is its Seif. matrix

Ex: Trefoil revisited : $V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Let $\omega \in \mathbb{C}$ be an n^{th} root of unity.
Then

$$(1-\omega)V + (1-\bar{\omega})V^T = \begin{bmatrix} 2-(\omega+\bar{\omega}) & 1-\omega \\ 1-\bar{\omega} & 2-(\omega+\bar{\omega}) \end{bmatrix}$$

Let $\omega = a+bi$, $\bar{\omega} = a-bi$. Note that $a^2+b^2 = 1$ & $a,b \leq 1$

$$(1-\omega)V + (1-\bar{\omega})V^T = \begin{bmatrix} 2-2a & 1-\omega \\ 1-\bar{\omega} & 2-2a \end{bmatrix}$$

Eigenvalues: $(2-2a-\lambda)^2 - (1-\omega)(1-\bar{\omega}) = 0$

$$\lambda^2 - 2(2-2a)\lambda + (2-2a)^2 - 1 + \omega + \bar{\omega} - 1 = 0$$

$$\lambda^2 - (4-4a)\lambda + 4a - 6a + 2 = 0$$

$$\lambda = \frac{4-4a \pm \sqrt{(4-4a)^2 - 4(4a^2-6a+2)}}{2}$$

$$= 2-2a \pm \sqrt{2-2a} > 0$$

$$\Rightarrow \sigma_\omega(K) = 2 \quad \forall \omega.$$

If $a=-1$ (so $b=0$ & $\omega=-1$), $\lambda = 4 \pm 2 > 0$ (same calculation as previous example)

$$\Rightarrow \sigma_{-1}(K) = 2 = \sigma(K)$$

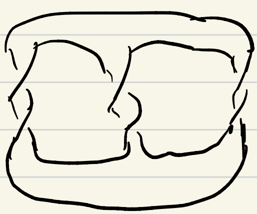
Nullity: $\eta_\omega(K) = 0 + 1 - 1 = 0 \quad \forall \omega.$

Thm: If $\det L$ is not a perfect square, then L is not χ-slice

Thm (Donald-Owens):
Let L be an oriented link with $\eta_\omega(L)=0$ for some ω
If L bounds an oriented surface with $\chi=1$, then
$\sigma_\omega(L)=0$.

Note: this only obstructs L from bounding oriented surfaces with $\chi=1$. Often we can do more.

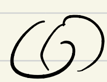
Ex: $L=P(2,2,2)$



$\sigma_{-1}(L)\neq 0$
$\eta_{-1}(L)=0$ for each chosen orientation (check)

So by thm, L doesn't bound an oriented surface

If L bounds a nonorientable surface F w/ $\chi=1$,
then $F=$ Mobius $\sqcup$ Mobius $\sqcup$ Disk (check)

If we remove one of the Mobius bands, then we have a new surface $F=$ Mobius $\sqcup$ Disk bounded by  Hopf link. But, we know Hopf link is not χ-slice since $\det L\neq 0$

$\Rightarrow L$ not χ-slice