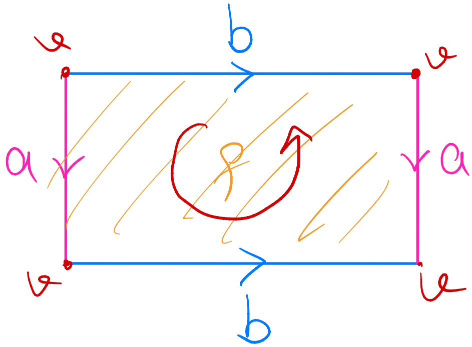
HW Solutions

Intro. to Homology

1 T^2



$$\boxed{H_i(X) = \ker(d_{i-1}) / \operatorname{Im}(d_i)}$$

$$\cdots \longrightarrow 0 \xrightarrow{d_2} \underbrace{C_2}_{\langle f \rangle} \xrightarrow{d_1} \underbrace{C_1}_{\langle a,b \rangle} \xrightarrow{d_0} \underset{\langle v \rangle}{\overset{}{C_0}} \xrightarrow{d_{-1}} 0 \longrightarrow \cdots$$

$$C_0 = \langle v \rangle$$

$$a \overset{d_0}{\longmapsto} v - v = 0$$

$$b \overset{d_0}{\longmapsto} v - v = 0$$

$$f \longmapsto a + b - a - b = 0$$

$$H_i(X) = \ker(d_{i-1}) / \operatorname{Im}(d_i) = 0 \qquad \forall i < 0$$

$$H_0(X) = \ker(d_{-1}) / \operatorname{Im}(d_1) = \langle v \rangle / 0 = \langle v \rangle \cong \mathbb{Z}$$

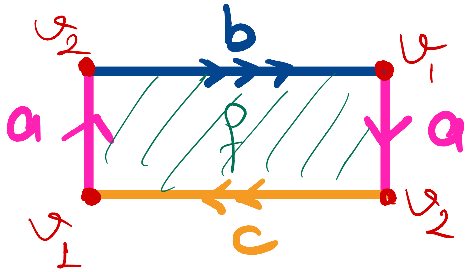
$$H_1(X) = \ker(d_0) / \operatorname{Im}(d_1) = \langle a, b \rangle / 0 = \langle a, b \rangle = \mathbb{Z} \oplus \mathbb{Z}$$

$$H_2(X) = \ker(d_1) / \operatorname{Im}(d_2) = \langle f \rangle / 0 = \langle f \rangle \cong \mathbb{Z}$$

$$H_i(X) = \ker(d_{i-1}) / \operatorname{Im}(d_i) = 0/0 = 0 \quad \forall i \geq 3$$

$$\Longrightarrow H_i(T^2) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } i = 1 \\ 0 & \text{Otherwise} \end{cases}$$

2 MB



$$\cdots \longrightarrow 0 \xrightarrow{d_2} C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0} C_0 \xrightarrow{d_{-1}} 0 \longrightarrow \cdots$$

$$C_2 = \langle f \rangle \qquad C_1 = \langle a, b, c \rangle \qquad C_0 = \langle v_1, v_2 \rangle$$

$$a \longmapsto v_2 - v_1$$

$$b \longmapsto v_1 - v_2$$

$$c \longmapsto v_1 - v_2$$

$$f \longmapsto a + b + a + c = 2a + b + c$$

$$H_i(MB) = \ker(d_{i-1}) \big/ \mathrm{Im}(d_i) = 0/0 = 0 \quad \forall\, i < 0$$

$$H_0(MB) = \ker(d_{-1}) \big/ \mathrm{Im}(d_0) = \langle v_1, v_2 \rangle \big/ \langle v_2 - v_1, v_1 - v_2 \rangle$$

$$= \langle v_1, v_2 \rangle \big/ \langle v_2 - v_1 \rangle = \langle v_1, \cancel{v_2 - v_1} \rangle \big/ \langle \cancel{v_2 - v_1} \rangle = \langle v_1 \rangle = \mathbb{Z}$$

$$H_1(MB) = \ker(d_0) \big/ \operatorname{Im}(d_1)$$

$$= \langle a+b,\ a+c,\ \underbrace{b-c}_{a+b-(a+c)} \rangle \big/ \langle 2a+b+c \rangle$$

$$= \langle a+b,\ a+c \rangle \big/ \langle 2a+b+c \rangle$$

$$= \langle a+b,\ \cancel{2a+b+c} \rangle \big/ \langle \cancel{2a+b+c} \rangle$$

$$= \langle a+b \rangle \cong \mathbb{Z}$$

$$H_i(MB) = \ker(d_{i-1}) \big/ \operatorname{Im}(d_i) = 0/0 = 0 \quad \forall i \geq 2$$

$$H_i(MB) = \begin{cases} \mathbb{Z} & i = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

3 Mayer–Vietoris Thm:

$$X = A \cup B \qquad (A, B \underset{\text{open}}{\subseteq} X)$$

$$\Rightarrow \quad \cdots \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(X)$$

$$\to H_{k-1}(A \cap B) \to H_{k-1}(A) \oplus H_{k-1}(B) \to H_{k-1}(X)$$

$$\cdots \to H_0(X) \to 0 \qquad \text{is exact.}$$

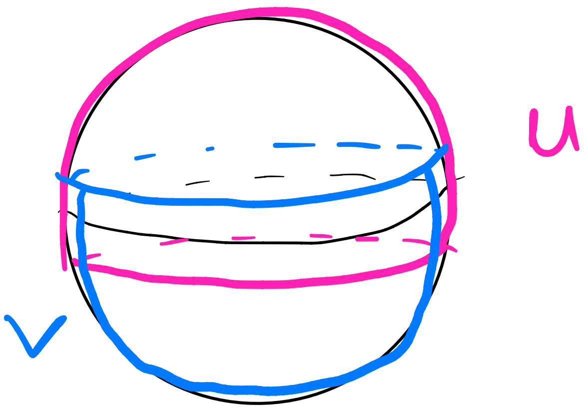
exact :=
$$\to A_3 \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_0} A_0 \to \cdots$$

$$\mathrm{Im}(f_i) = \ker(f_{i-1})$$

Moreover $\quad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

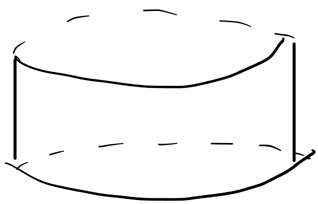
$$\Rightarrow \quad C \cong B/\mathrm{Im}(f) = B/\ker(g)$$

A, B, C abelian $\Rightarrow B \cong A \oplus C$



$A, B \subseteq S^2$ open

$$S^2 = A \cup B$$

$A \cap B =$  $S^1 \times I \underset{h.e.}{\simeq} S^1$

$$\Longrightarrow H_k(A \cap B) = \mathbb{Z} \quad \text{for} \quad i = 0, 1$$

$$H_k(A \cap B) = 0 \quad \forall i \neq 0, 1$$

$A, B \underset{h.e}{\simeq} \{pt\} \quad \therefore \quad H_0 = \mathbb{Z}$

$$H_i = 0 \quad \forall i \neq 0$$

$$\cdots \to \underbrace{H_3(\underbrace{A\cap B}_{S^1})}_{0} \to \underbrace{H_3(A)}_{0} \oplus \underbrace{H_3(B)}_{0} \to H_3(S^2)$$

$$\xrightarrow{g_2} \underbrace{H_2(\underbrace{A\cap B}_{S^1})}_{0} \xrightarrow{f_2} \underbrace{H_2(A)}_{0} \oplus \underbrace{H_2(B)}_{0} \xrightarrow{d_2} H_2(S^2)$$

$$\xrightarrow{g_1} \underbrace{H_1(\underbrace{A\cap B}_{S^1})}_{\mathbb{Z}} \xrightarrow{f_1} \underbrace{H_1(A)}_{0} \oplus \underbrace{H_1(B)}_{0} \xrightarrow{d_1} H_1(S^2)$$

$$\xrightarrow{g_0} \underbrace{H_0(\underbrace{A\cap B}_{S^1})}_{\mathbb{Z}} \xrightarrow{f_0} \underbrace{H_0(A)}_{\mathbb{Z}} \oplus \underbrace{H_0(B)}_{\mathbb{Z}} \xrightarrow{d_0} H_0(S^2) \to 0$$

$$c \mapsto (c, c)$$

$$\to \mathbb{Z} \xrightarrow{f_0} \mathbb{Z} \oplus \mathbb{Z} \to H_0(S^2) \to 0$$

$$H_0(S^2) = \mathbb{Z} \oplus \mathbb{Z} / \operatorname{Im}(f_0)$$

$$= \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z} = \mathbb{Z}$$

$$0 \xrightarrow{d_1} H_1(S^2) \xrightarrow{g_0} \mathbb{Z} \xrightarrow{f_0} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

$$c \longmapsto (c, c)$$

f_0 is inj.

$$\ker(f_0) = 0$$

$$\operatorname{Im}(g_0) = \ker(f_0) = 0$$

$$H_1(S^2) = \operatorname{Im}(g_0) / \ker(d_1) = 0$$

$$\underbrace{H_2(A)}_{0} \oplus \underbrace{H_2(B)}_{0} \xrightarrow{\alpha_2} H_2(S^2) \xrightarrow{g_1} \underbrace{H_1(\underbrace{A \cap B}_{S^1})}_{\mathbb{Z}} \xrightarrow{f_1} \underbrace{H_1(A)}_{0} \oplus \underbrace{H_1(B)}_{0}$$

$$0 \to H_2(S^2) \xrightarrow[\cong]{} \mathbb{Z} \to 0$$

$$H_2(S^2) = 0 \oplus \mathbb{Z} = \mathbb{Z}$$

For $i \geq 2$

$$0 \to \underbrace{H_{i+1}(A)}_{0} \oplus \underbrace{H_{i+1}(B)}_{0} \to H_{i+1}(S^2) \xrightarrow{g_2} \underbrace{H_i(\underbrace{A \cap B}_{S^1})}_{0} \to 0$$

$$\Rightarrow H_{i+1}(S^2) = H_{i+1}(A) \oplus H_{i+1}(B) \quad H_i(A \cap B)$$

$$\Rightarrow H_k(S^2) = 0 \quad \forall k \geq 3$$

So,

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

4 Calculate the Homology groups of a 4-man. with

- only 1 0-handle and
- some number of 2-handles.

Solution:

In dim = 4 $\quad$ $0\text{-h} \cup 2\text{-h} \underset{\text{h.e.}}{\sim} S^2$

homotopy equiv.

$$0\text{-h} \cup 2\text{-h} = \underbrace{(D^0 \times D^4)}_{D^4} \cup (D^2 \times D^2)$$

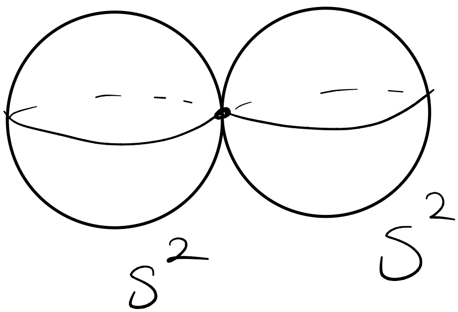
$$\cong S^2 \times D^2 \sim S^2$$

if the framing is 0.

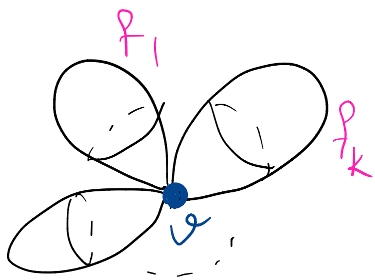
but even if the framing is NOT 0 still works.

$0\text{-h} \cup (2\text{-h}) \cup (2\text{-h})$

$= (D^0 \times D^4) \cup (D^2 \times D^2) \cup (D^2 \times D^2)$

$=$ 

More generally,

$0\text{-h} \bigcup_k (2\text{-h}) =$ 

wedge of k-spheres

$$\cdots \xrightarrow{d_3} \underbrace{C_3(X)}_{0} \xrightarrow{d_2} \underbrace{C_2(X)}_{\substack{\langle f_1, f_2, \ldots, f_k \rangle \\ \| \\ \mathbb{Z}^k}} \xrightarrow{d_1} \underbrace{C_1(X)}_{0} \xrightarrow{d_0} \underbrace{C_0(X)}_{\substack{\langle v \rangle \\ \| \\ \mathbb{Z}}} \xrightarrow{d_{-1}} 0$$

$$\boxed{H_i(X) = \ker(d_{i-1}) / \operatorname{Im}(d_i)}$$

$$H_i(X) = 0 \qquad \forall i < 0$$

$$H_0(X) = \ker(d_{-1}) / \operatorname{Im}(d_0) = \langle v \rangle / 0 = \mathbb{Z}$$

$$H_1(X) = \ker(d_0) / \operatorname{Im}(d_1) = 0/0 = 0$$

$$H_2(X) = \ker(d_1) / \operatorname{Im}(d_2) =$$

$$= \langle f_1, f_2, \ldots, f_k \rangle / 0 = \mathbb{Z}^k$$

$$H_3(X) = \ker(d_2) / \operatorname{Im}(d_1) = 0/0 = 0$$

$$H_i(X) = 0 \qquad \forall\, i \geq 3$$

$$\Rightarrow H_i(X) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^k & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

If we add a 4-h

then $H_4(X) \cong \mathbb{Z}$