HW Solutions

Intersection Form

Problems

1. Find the intersection form of

a. S^4

b. $\mathbb{CP}^2$

c. $S^2 \times S^2$

d. $k\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2$

Solutions

(a) S^4

$H_2(S^4) = 0$

$\chi_{S^4} = 0$

(b) $\mathbb{CP}^2$

$$H_2(\mathbb{CP}^2; \mathbb{Z}) = \langle [L] \rangle \cong \mathbb{Z}$$

$L \subseteq \mathbb{CP}^2$: proj. line repr. a generator

Any two complex line intersects exactly at one point.

i.e. $[L][L] = 1$

$$\Longrightarrow Q_{\mathbb{CP}^2} = \langle 1 \rangle$$

Remark: Similarly $H_2(\overline{\mathbb{CP}^2}; \mathbb{Z}) = \langle [\bar{L}] \rangle$

$[\bar{L}][\bar{L}] = -1$

$$Q_{\overline{\mathbb{CP}^2}} = \langle -1 \rangle$$

(c) $S^2 \times S^2$

$$H_2(S^2 \times S^2; \mathbb{Z}) = \langle [\alpha], [\beta] \rangle = \mathbb{Z} \oplus \mathbb{Z}$$

where $[\alpha] = [S^2 \times \{pt\}]$ $\quad [\beta] = [\{pt\} \times S^2]$

$[\alpha]^2 = 0 \qquad Q_{S^2 \times S^2}([\alpha], [\alpha]) = 0$

$[\beta]^2 = 0 \qquad Q_{S^2 \times S^2}([\beta], [\beta]) = 0$

$$[\alpha][\beta] = 1 = [\beta][\alpha]$$

$$\Longrightarrow Q_{S^2 \times S^2}([\alpha], [\beta]) = 1 = Q_{S^2 \times S^2}([\beta], [\alpha])$$

$$Q_{S^2 \times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

d) $k\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2$

$Q_{\mathbb{CP}^2} = \langle 1 \rangle \qquad Q_{\overline{\mathbb{CP}}^2} = \langle -1 \rangle$

$$\Rightarrow Q_{k\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2}$$

$$= k\, Q_{\mathbb{CP}^2} \oplus m\, Q_{\overline{\mathbb{CP}}^2}$$

$$= k \langle 1 \rangle \oplus m \langle -1 \rangle$$

② Show that there is NOT a closed, smooth, orientable man. X with $Q_X = E_8$.

Solution:

Suppose X is a closed, smooth, orientable man. with $Q_X = E_8$. Assuming $\pi_1(X) = 1$

$$\xRightarrow[\text{Rochlin Thm}]{} \sigma(X) \equiv 0 \pmod{16}$$

$\rightarrow\leftarrow$

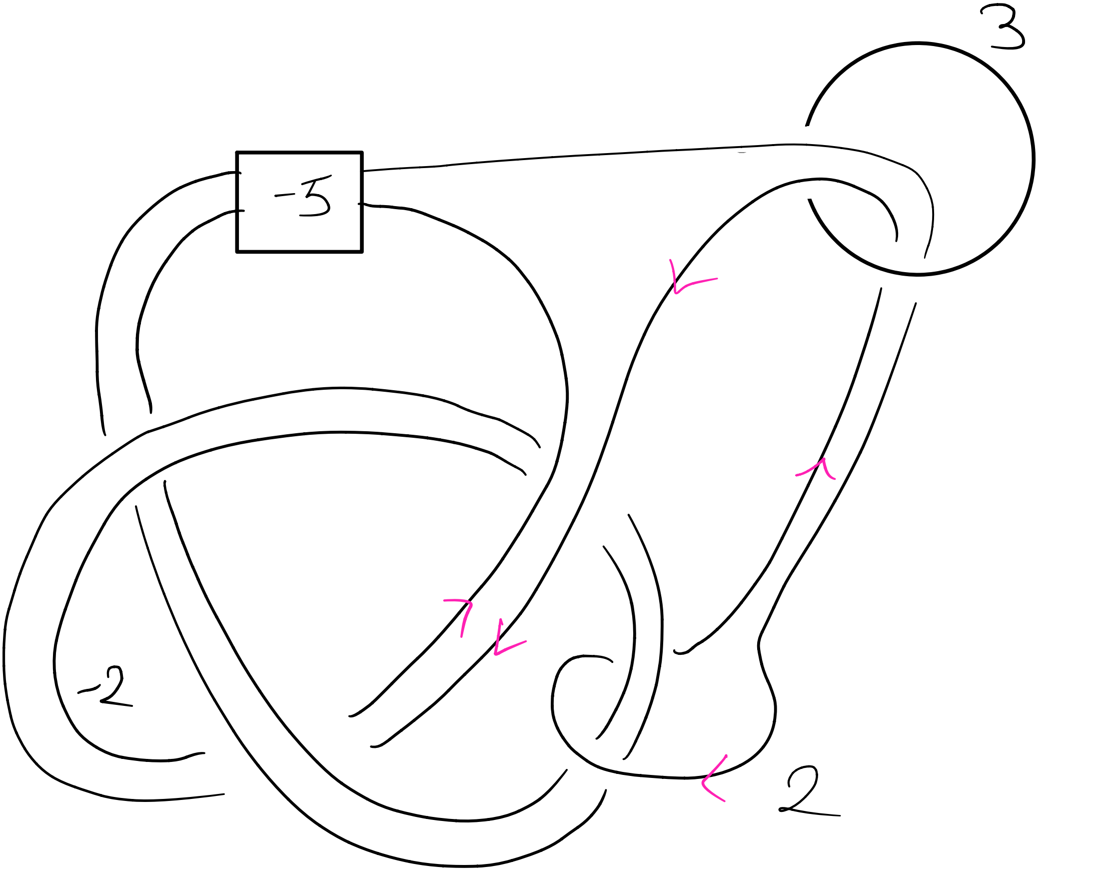
$$\sigma(X) = 8$$

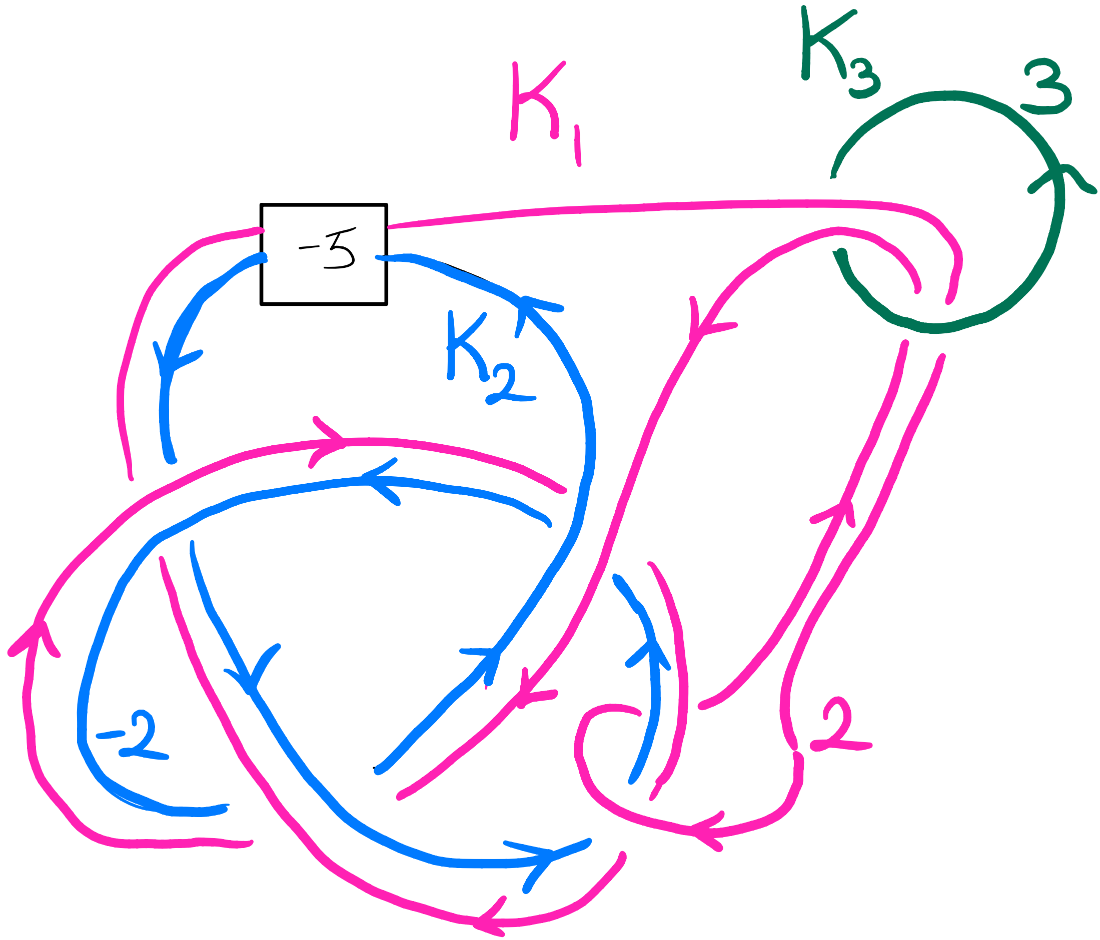
$$E_8 = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & 0 & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ 0 & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$$

unimodular
even
pos. definite
$\operatorname{rank}(E_8) = 8 = \sigma(E_8)$

Rochlin's Thm. has been extended to arbitrary fundamental groups by Ozsvath–Szabo.

3. Find the intersection form of the following diagram.

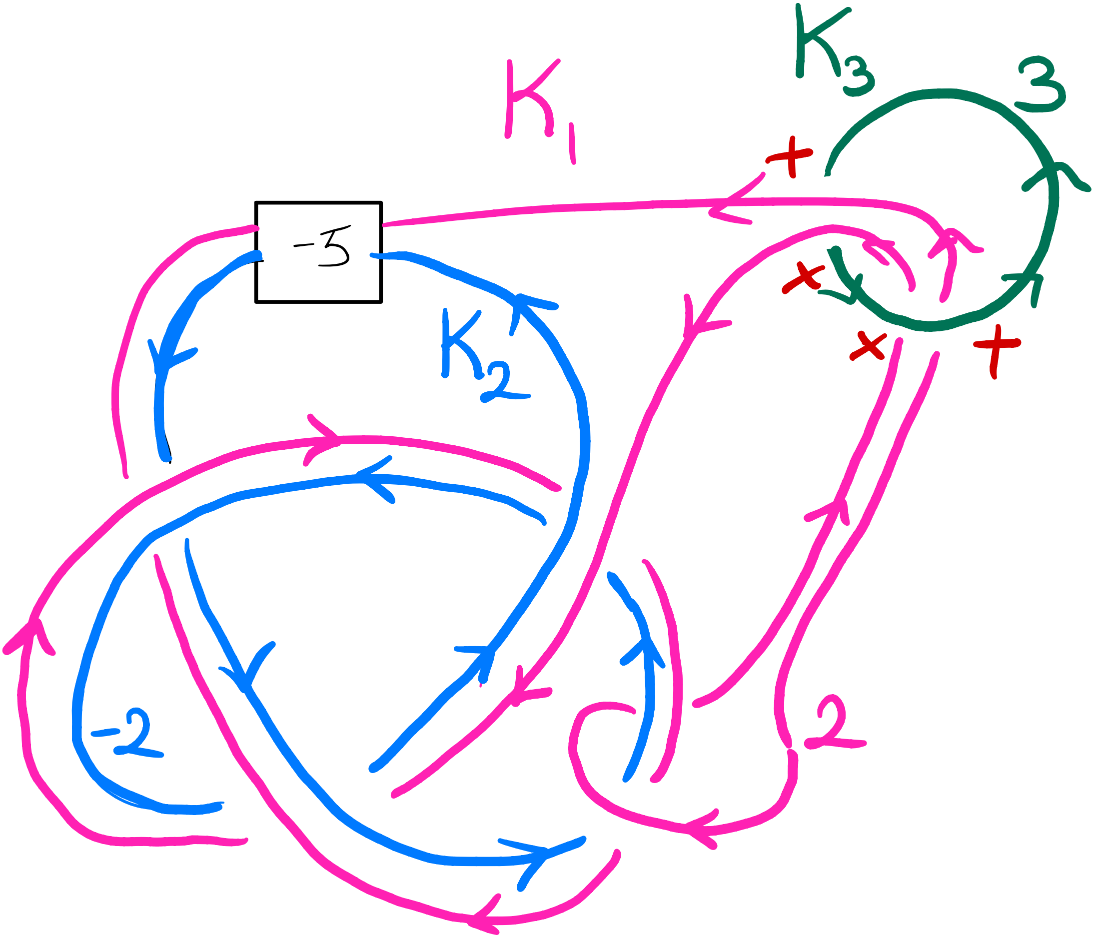




$$Q_X([K_1],[K_1]) = \#(S_1,\tilde{S}_1) = 2$$

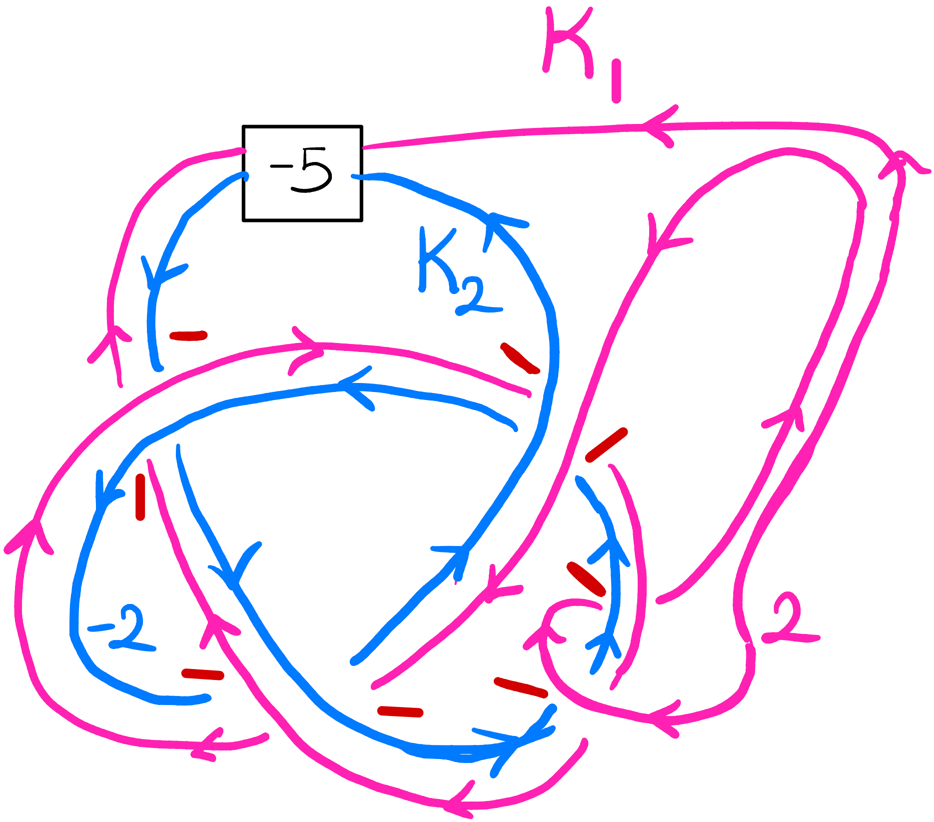
$$Q_X([K_2],[K_2]) = \#(S_2,\tilde{S}_2) = -2$$

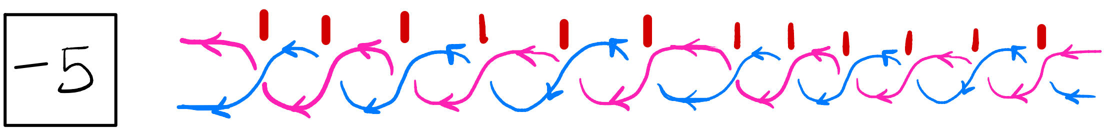
$$Q_X([K_3],[K_3]) = \#(S_3,\tilde{S}_3) = 3$$



$$lk(K_1, K_3) = \frac{4-0}{2} = 2$$

$$lk(K_2, K_3) = 0$$





$$\#(K_1, K_2) = \frac{0 - 20}{2} = -10$$

$$lk(K_1, K_2) = -10$$

$$\Rightarrow Q_x = \begin{array}{c|ccc|c} & K_1 & K_2 & K_3 & \\ & \begin{bmatrix} 2 & -10 & 2 \\ -10 & -2 & 0 \\ 2 & 0 & 3 \end{bmatrix} & & & \begin{array}{c} K_1 \\ K_2 \\ K_3 \end{array} \end{array}$$