Solutions

1.) Yes! $\varphi(f_1) = e_1 - e_2$, $\varphi(f_2) = e_2 - e_3$

2&3.) <u>Answer</u>: $\exists$ lattice embedding $\iff$ $n = -ma^2 + (a+1)^2$ for some $a \neq 0$

<u>Proof</u>:

Let $\varphi : (\mathbb{Z}^{m+1}, Q) \longrightarrow (\mathbb{Z}^{m+1}, -I)$ be a lattice embedding.

Set $v_i = \varphi(f_i)$ $\forall\ 1 \le i \le m+1$

and $v_i \cdot v_j = -I(v_i, v_j)$ (negative dot product)

Then, by definition,

$$v_1 \cdot v_1 = -n$$

$$v_i \cdot v_i = -2 \quad \forall\ 2 \le i \le m+1$$

$$v_i \cdot v_j = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| \neq 1 \end{cases}$$

<u>Case I</u>: $m = 1$, $Q = \begin{bmatrix} -n & 1 \\ 1 & -2 \end{bmatrix}$

Let $v_2 = x_1 e_1 + x_2 e_2$. Then $-2 = v_1 \cdot v_1 = -x_1^2 - x_2^2 \Rightarrow x_1 = \pm 1$, $x_2 = \pm 1$

Up to changing the basis, we may assume $x_1 = 1$, $x_2 = -1$

So $v_2 = e_1 - e_2$.

Next let $v_1 = y_1 e_1 + y_2 e_2$

Then $v_1 \cdot v_2 = 1 \Rightarrow -x_1 y_1 - x_2 y_2 = 1 \Rightarrow y_1 - y_2 = 1 \Rightarrow y_1 = y_2 + 1$

Thus $-n = v_1 \cdot v_1 = -y_1^2 - y_2^2 = -(y_2+1)^2 - y_2^2$

Hence $\exists$ lattice embedding $\Leftrightarrow$ $n = a^2 + (a+1)^2$ for some $a \geq 1$.

<u>Case 2</u>: $m = 2$, $Q = \begin{bmatrix} -n & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

Let $v_3 = x_1 e_1 + x_2 e_2 + x_3 e_3$. Then $v_3 \cdot v_3 = -2 \Rightarrow -x_1^2 - x_2^2 - x_3^2 = -2$

$\Rightarrow (x_1, x_2, x_3) \in \{(\pm 1, \pm 1, 0), (0, \pm 1, \pm 1), (\pm 1, 0, \pm 1)\}$.

Up to changing basis, we may assume $v_3 = e_1 - e_2$

Similarly, $v_2 = \pm e_i \pm e_j$ for some $i \neq j \in \{1, 2, 3\}$.

If $i = 1$ and $j = 2$ (or $i = 2$ and $j = 1$), then $v_2 \cdot v_3 \in \{0, \pm 2\}$,

which contradicts the fact that $v_2 \cdot v_3 = 1$.

Hence we may assume $i = 2$ and $j = 3$

Up to a change of basis, we have $v_2 = e_2 - e_3$

Let $v_1 = y_1 e_1 + y_2 e_2 + y_3 e_3$.

Since $v_1 \cdot v_3 = 0$, we have $y_2 - y_1 = 0 \Rightarrow y_1 = y_2$

Since $v_1 \cdot v_2 = 1$, we have $y_3 - y_2 = 1 \Rightarrow y_3 = y_2 + 1$

Finally, $n = -v_1 \cdot v_1 = y_1^2 + y_2^2 + y_3^2 = 2y_1^2 + y_3^2$

Thus $\exists$ embedding iff $n = 2a^2 + (a+1)^2$, where $a \neq 0$

(if $a = 0$, $n = 1$, which is a contradiction)

Case 3: $m \geq 3$

As in Case 2, we may assume $v_{m+1} = e_1 - e_2$, $v_m = e_2 - e_3$

Now, since $v_{m-1} \cdot v_{m-1} = -2$, again we have $v_{m-1} = \pm e_i \pm e_j$, $i \neq j$.
Assume $i = 2$, then since $v_{m-1} \cdot v_{m+1} = 0$, we must have
$j = 1$ and $v_{m-1} = \pm(e_1 + e_2)$. Since $v_{m-1} \cdot v_{m-2} = 1$,
we have that $v_{m-2} = \mp e_1 + \cdots$ or $v_{m-2} = \mp e_2 + \cdots$
Since $v_{m-2} \cdot v_{m+1} = 0$, we then have $v_{m-2} = \mp(e_1 + e_2) + \cdots$
But then $v_{m-2} \cdot v_{m-1} \neq 1$, which is a contradiction.
Thus $i \neq 2$. Similarly, $i \neq 1$ and $j \neq 1, 2$.

Hence $i = 3$ (since $v_{m-1} \cdot v_m = 1$). and we may assume $j = 4$
So $v_{m-1} = e_3 - e_4$.

We have: $v_{m+1} = e_1 - e_2$, $v_m = e_2 - e_3$, $v_{m-1} = e_3 - e_4$

Continuing in this way (inductively)
we may assume that $v_{m-i} = e_{i+2} - e_{i+3}$ $\forall\ 1 \leq i \leq m-2$

Let $v_1 = \sum_{i=1}^{m+1} y_i e_i$

$v_1 \cdot v_{m+1} = 0 \Rightarrow y_2 - y_1 = 0 \Rightarrow y_2 = y_1$

$v_1 \cdot v_m = 0 \Rightarrow y_3 - y_2 = 0 \Rightarrow y_3 = y_2$

$v_1 \cdot v_{m-i} = 0 \Rightarrow y_{i+3} - y_{i+2} = 0 \Rightarrow y_{i+3} = y_{i+2}$ $\forall\ 1 \leq i \leq m-2$

and

$v_1 \cdot v_2 = 1 \Rightarrow y_{m+1} - y_m = 1 \Rightarrow y_{m+1} = y_m + 1$

Hence $n = -v_1 \cdot v_1 = -y_1^2 - y_2^2 - \cdots - y_{m+1}^2 = -m y_1^2 - (y_1 + 1)^2$

$\Rightarrow \exists$ embedding $\iff n = m a^2 + (a+1)^2$ for $a \neq 0$.