# Solutions

1.) <u>Claim</u>: $\exists$ a cubiquitous lattice embedding $\Leftrightarrow$ $n=5$.

<u>proof</u>:

Recall $\exists$ lattice embedding if and only if

$n = a^2 + (a+1)^2$ for some $a \geq 1$.

Such a lattice embedding is given by

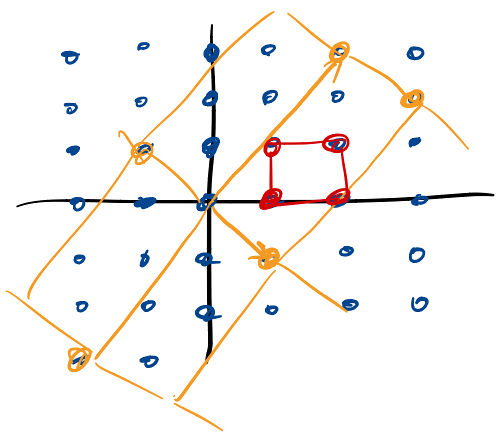
$\varphi(f_1) = ae_1 + (a+1)e_2$

$\varphi(f_2) = e_1 - e_2$

We already showed that if $a=1$ (so $n=5$), then $\varphi$ is cubiquitous.

We now show that if $a>1$, $\varphi$ is not cubiquitous.

<u>Ex</u>: $a=2$



Let $C = \{0,1\}^2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We will show that $\operatorname{Im}\varphi \cap C = \phi$.

Equivalently, we will show that
$$ax + y = \lambda_1 + 1$$
$$(a+1)x - y = \lambda_2$$
has no integer solution $\forall \ \lambda_1, \lambda_2 \in \{0,1\}$.

Solving the system over $\mathbb{R}$ gives:

$$\left[\begin{array}{cc|c} a & 1 & \lambda_1+1 \\ a+1 & -1 & \lambda_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{1+\lambda_1+\lambda_2}{2a+1} \\ 0 & 1 & \frac{-a\lambda_2+a\lambda_1+a+\lambda_1+1}{2a+1} \end{array}\right]$$

Hence the system has a unique solution:

$$x = \frac{1+\lambda_1+\lambda_2}{2a+1}$$

$$y = \frac{-a\lambda_2+a\lambda_1+a+\lambda_1+1}{2a+1}$$

Since $a \geq 2$ and $\lambda_1, \lambda_2 \in \{0,1\}$, $\quad 1+\lambda_1+\lambda_2 < 3 < 2a+1$

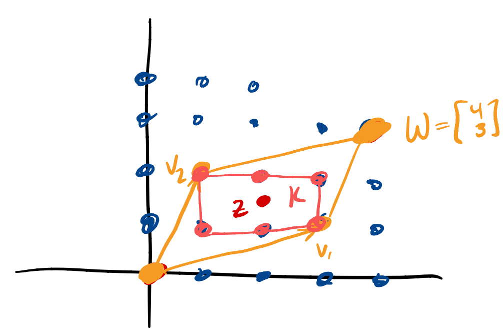
Thus $x \notin \mathbb{Z} \Rightarrow \varphi$ is not cubiquitous.

2) Let $z = \frac{1}{2}w$.

For a unit cube $C$, let $\bar{C}$ denote the solid unit cube.

Let $K = \{\text{unit cubes } C \mid z \in \bar{C}\}$

Let $\Lambda$ be the lattice generated by $S$.




Note: • If $x \in K$, then $w-x \in K$
• If $x \in \Lambda$, then $w-x \in \Lambda$

Hence $x \in K \cap \Lambda \iff w-x \in K \cap \Lambda$

Let $x \in K$. What is the angle between $x$ and $w-x$?

$$x \in K \Rightarrow x = \sum_{i \in R_o} \frac{k_i+\varepsilon_i}{2} e_i + \sum_{i \in R_e} \frac{k_i+2\varepsilon_i}{2} e_i + \sum_{i \in O} \varepsilon_i e_i \quad \text{for } \varepsilon_i \in \{-1,1\}.$$

$$\Rightarrow W - x = \sum_{i \in R_o} \frac{k_i - \varepsilon_i}{2} e_i + \sum_{i \in R_e} \frac{k_i - 2\varepsilon_i}{2} e_i - \sum_{i \in O} \varepsilon_i e_i$$

Thus
$$\langle x, W - x \rangle = \sum_{i \in R_o} \frac{k_i^2 - \varepsilon_i^2}{4} + \sum_{i \in R_e} \frac{k_i^2 - 4\varepsilon_i^2}{4} e_i - \sum_{i \in O} \varepsilon_i^2$$
$$= \sum_{i \in R_o} \frac{k_i^2 - 1}{4} + \sum_{i \in R_e} \frac{k_i^2 - 4}{4} - |O|$$
$$\geq 0 \quad \text{since} \quad \sum_{i \in R_o} \frac{k_i^2 - 1}{4} + \sum_{i \in R_e} \frac{k_i^2 - 4}{4} \geq |O|$$

$\Rightarrow$ The angle between $x$ and $W - x$ is at most $90^\circ$

Now let $y \in \Lambda$. What is the angle between $y$ and $W - y$?

$y \in \Lambda \Rightarrow y = \sum_{i=1}^{n} y_i v_i$ for some $y_i \in \mathbb{Z}$ and $W - y = \sum_{i=1}^{n} (1 - y_i) v_i$.

$$\langle y, W - y \rangle = \sum_{i=1}^{n} y_i (1 - y_i) a_i - \sum_{i=1}^{n-1} y_i (1 - y_{i+1}) - \sum_{i=1}^{n-1} y_{i+1} (1 - y_i) \quad (\text{since } S \text{ is standard})$$
$$= y_1 (1 - y_1)(a_1 - 1) + \sum_{i=2}^{n-2} y_i (1 - y_i)(a_i - 2) + y_n (1 - y_n)(a_n - 1) - \sum_{i=1}^{n-1} (y_i - y_{i+1})^2$$
$$\leq 0 \quad (\text{since } a_i \geq 2 \ \forall i \text{ and } y_i (1 - y_i) \leq 0 \ \forall i)$$

$\Rightarrow$ The angle between $y$ and $W - y$ is at least $90^\circ$

Note that $\langle y, W - y \rangle = 0 \iff$ each term in the sum is $0$
$\iff y_i = 1 \ \forall i$ or $y_i = 0 \ \forall i$
$\iff y = 0$ or $y = W$.

Hence if $y \neq 0, W$, then the angle between $y$ and $W - y$ is greater than $90^\circ$.

If $\sum_{i \in R_o} \frac{k_i^2-1}{4} + \sum_{i \in R_e} \frac{k_i^2-4}{4} > |O|$, the have that $\langle x, w-x \rangle < 0$ $\forall$ $x \in K$ and $\langle y, w-y \rangle \geq 0$ $\forall$ $y \in \Lambda$

It follows that $K \cap \Lambda = \emptyset$ $\Rightarrow$ $\Lambda$ is not cubiquitous.

If $\sum_{i \in R_o} \frac{k_i^2-1}{4} + \sum_{j \in R_e} \frac{k_j^2-4}{4} = |O|$ and $|R_e| \geq 2$.

then $\langle x, w-x \rangle \leq 0$ $\forall$ $x \in K$ and $\langle y, w-y \rangle \geq 0$ $\forall y \in \Lambda$ with equality if and only if $y \in \{0, w\}$.

Note that $K$ has $2^{|R_e|+|O|}$ unit cubes. Since $|R_e| \geq 2$, it follows that $\exists$ $C \subset K$ such that $0, w \notin C$.

Consequently $C \cap \Lambda = \emptyset$. $\Rightarrow$ $\Lambda$ is not cubiquitous. $\blacksquare$

3) Answer: $\exists$ cubiquitous embedding iff $n = m+4$ or $n = m$.

We know that $\exists$ lattice embedding if and only if $n = ma^2 + (a+1)^2$ for some $a \neq 0$.

Up to changing basis, it is given by:

$$\varphi(f_1) = \sum_{i=1}^{m} a e_i + (a+1) e_{m+1}$$
$$\varphi(f_2) = e_m - e_{m+1}$$
$$\vdots$$
$$\varphi(f_{m+1}) = e_1 - e_2$$

Let $S = \{\varphi(f_1), \dots, \varphi(f_{m+1})\}$. Then $S$ is a standard subset and the Wu element of $S$ is $W = (a+1)e_1 + \sum_{i=2}^{m+1} a e_i$

Assume $a \neq -2, -1, 1$

- If $a$ is even, then (using the notation in previous problem),

$$\sum_{i \in R_0} \frac{k_i^2 - 1}{4} + \sum_{i \in R_2} \frac{k_i^2 - 4}{4} = \frac{a^2 + 2a}{4} + \frac{m(a^2 - 4)}{4} > 0 = |\sigma|$$

- If $a$ is odd, then

$$\sum_{i \in R_0} \frac{k_i^2 - 1}{4} + \sum_{i \in R_2} \frac{k_i^2 - 4}{4} = \frac{m(a^2 + 2a)}{4} + \frac{a^2 - 4}{4} > 0 = |\sigma|$$

In either case, by problem 2, $\varphi$ is not cubiquitous.

Now suppose $a = -2$. Then $\sum_{i \in R_0} \frac{k_i^2 - 1}{4} + \sum_{i \in R_2} \frac{k_i^2 - 4}{4} = 0 = |\sigma|$

Since $w = -e_1 - \sum_{i=1}^{m+1} 2e_i$, we have $|R_2| \geq m$

If $m \geq 2$, then by problem 2, $\varphi$ is not cubiquitous.

If $m = 1$, then it is easy to see that $\varphi$ agrees (up to a change of basis) with the lattice embedding given in the $a = 1$, $m = 1$ case.

It remains to show that the embeddings are cubiquitous when $a = -1$ and $a = 0$.

this follows from topological reasons.