Solutions

1.) E_8 is easily seen to be positive definite (e.g. by checking the eigenvalues)
Hence if E_8 is diagonalizable, $\exists$ a lattice embedding $\varphi : (\mathbb{Z}^8, E_8) \to (\mathbb{Z}^8, I)$

Let $\{f_1, \ldots, f_8\}$ and $\{e_1, \ldots, e_8\}$ be the standard basis for $\mathbb{Z}^8$ for the domain and codomain, respectively.

As in the proof of the previous homework, up to a change of basis we have:

$$\varphi(f_1) = e_1 + e_2,\ \varphi(f_2) = e_2 + e_3,\ \ldots,\ \varphi(f_7) = e_7 + e_8$$

Let $\varphi(f_8) = \sum_{i=1}^{8} x_i e_i$.

Since $0 = Q(f_i, f_8) = \varphi(f_i) \cdot \varphi(f_8) = x_i + x_{i+1} \quad \forall\ i \in \{1,2,3,4,5,7\}$

we have $x_i = -x_{i+1} \quad \forall\ 1 \le i \le 5,\ i = 7$

$$\Rightarrow x_1 = -x_2 = x_3 = -x_4 = x_5 = -x_6 \quad \text{and} \quad x_7 = -x_8$$

Set $x := x_1$ and $y := x_7$.

Moreover, $1 = Q(f_6, f_8) = \varphi(f_6) \cdot \varphi(f_8) = x_6 + x_7 \Rightarrow x_7 = 1 - x_6 \Rightarrow y = 1 - x$

Finally, $2 = Q(f_8, f_8) = \varphi(f_8) \cdot \varphi(f_8) = \sum_{i=1}^{8} x_i^2 = 6x^2 + 2(1-x)^2$

$\Rightarrow 2 = 6x^2 + (1-x)^2$

$\Rightarrow x = 0$

$\Rightarrow \varphi(f_8) = e_7 + e_8$

$\Rightarrow \varphi(f_8) \cdot \varphi(f_7) = 2$, which is a contradiction.

Thus $\nexists$ a lattice embedding.

2.) Let $Q = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

Q is diagonalizable over $\mathbb{R}$ since
Any symmetric matrix is diagonalizable over $\mathbb{R}$

From homework 2, we know Q
not diagonalizable over $\mathbb{Z}$ since Q is negative
definite but $\nexists$ a lattice embedding $(\mathbb{Z}^2, Q) \to (\mathbb{Z}^2, -I)$.