Solutions

1.) a) The only surfaces with $\chi = 1$ and 3 boundary components are:

$D \sqcup D \sqcup \Sigma_1^1$

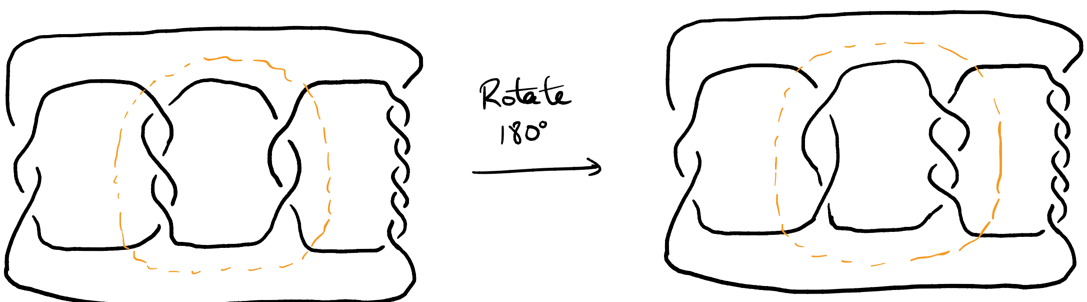
$D \sqcup D \sqcup P_2^1$

$D \sqcup M \sqcup M$

$D \sqcup A$

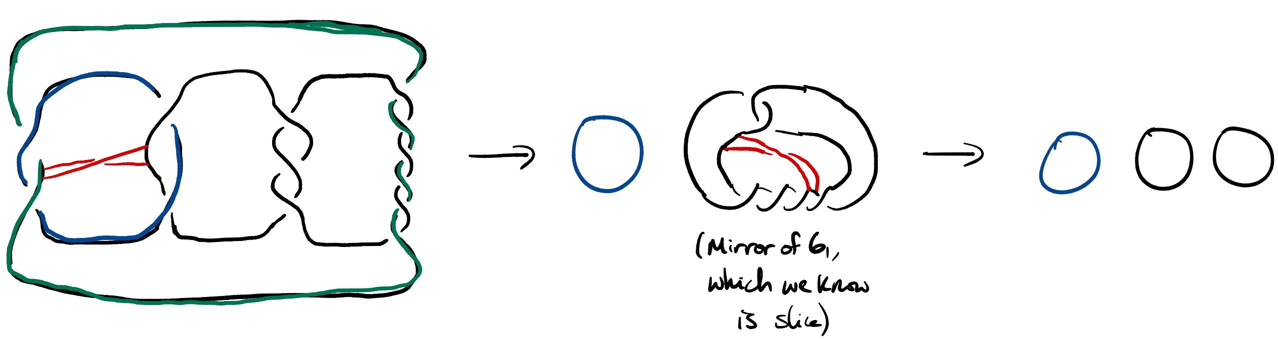
c) Read the proof of lemma 3.2 in "On χ-slice pretzel links."

d)

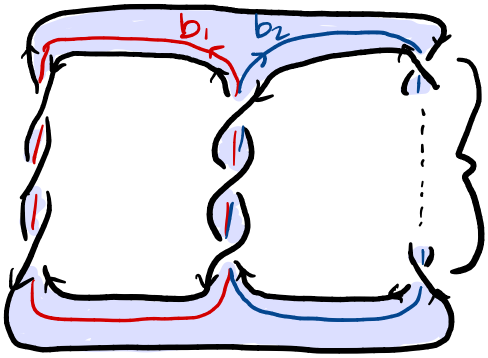


$P(2,-3,2,-6)$ $\qquad$ $P(2,2,-3,-6)$

We can see that $P(2,2,-3,-6)$ is χ-slice by using the following band moves:



3) a)



$$V = \begin{bmatrix} 3 & 3 \\ 0 & -6 \end{bmatrix}$$

$$V + V^T = \begin{bmatrix} 6 & 3 \\ 3 & -12 \end{bmatrix}$$

$\sigma(P(3,3,-15)) = 0$ (since eigenvalues are opposite sign)

$\det(P(3,3,-15)) = -81 = -9^2$

$\Rightarrow$ we cannot obstruct sliceness.

b) Let $Q = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & 1 \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & \\ & & 1 & & & -15 \end{bmatrix}$.

Using the usual notation, $\varphi(f_i) = e_i - e_{i+1}$ for $1 \leq i \leq 5$ and $\varphi(f_6) = \sum_{i=1}^{6} x_i e_i$. Since $\varphi(f_i) \cdot \varphi(f_6) = 0 \ \forall\, 1 \leq i \leq 4$, and $\varphi(f_5) \cdot \varphi(f_6) = 1$, we have:

$x_1 = x_2 = x_3 = x_4 = x_5$ and $x_6 = x_5 + 1$. Set $x := x_1$.

Since $\varphi(f_6) \cdot \varphi(f_6) = -15$, we have $\sum_{i=1}^{6} x_i^2 = 15$

$\Rightarrow 5x^2 + (x+1)^2 = 15 \Rightarrow 6x^2 + 2x - 14 = 0$.

But $6x^2 + 2x - 14 = 0$ has no integer solution, which is a contradiction.

thus $\nexists$ a lattice embedding and hence K is not slice

3) a) In class, we computed $V = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$. Thus

$$\Delta_K(t) = \det(V - tV^T) = \det\begin{bmatrix} -1+t & -t \\ 1 & -1+t \end{bmatrix} = (-1+t)^2 + t = t^2 - t + 1$$

c) $\Delta_K(t) = \det(V - tV^T) = \det\begin{bmatrix} 3-3t & 3 \\ -3t & -6+6t \end{bmatrix}$

$$= -18t^2 + 45t - 18 \doteq 18t^{-1} - 45 + 18t$$

If $\Delta_K(t) = f(t) \cdot f(t^{-1})$, then $f(t) = at + b$, $f(t^{-1}) = at^{-1} + b$

$$\Longrightarrow 18t^{-1} - 45 + 18t \doteq abt + (a^2 + b^2) + abt^{-1}$$

$$\Longrightarrow \begin{matrix} ab = -18 \\ a^2 + b^2 = 45 \end{matrix} \Longrightarrow \begin{matrix} a = 6 \\ b = -3 \end{matrix}$$

$$\Rightarrow \Delta_K(t) = (6t - 3)(6t^{-1} - 3)$$

Since $\Delta_K(t)$ splits, the Fox-Milnor theorem cannot be used to obstruct sliceness

Note: In this example, the lattice embedding obstruction was able to obstruct sliceness, while σ, det, and Δ were not. This doesn't always happen. There are examples of knots for which Δ or σ obstruct sliceness, but the lattice embedding does not.

However, it turns out that the lattice embedding obstruction is strictly stronger than det.