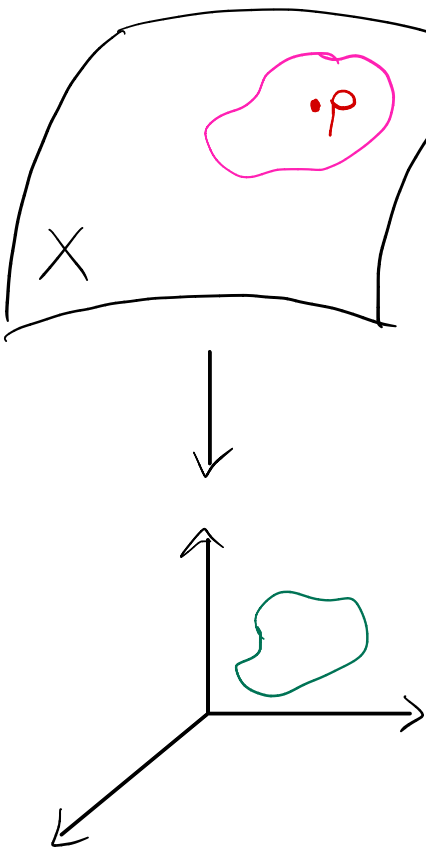
Classification of Surfaces and Euler Characteristic

n-dim. manifold:

A topological space that locally looks like $\mathbb{R}^n$.

Defn: A topological space X^n is called **n-dim. manifold** if every pt. has a nbhd which is homeomorphic to $\mathbb{R}^n$.



smooth man: if every pt. has a nbhd which is diffeomorphic to $\mathbb{R}^n$.

Examples

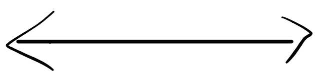
① 0-dim. man.s:

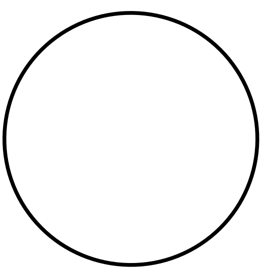
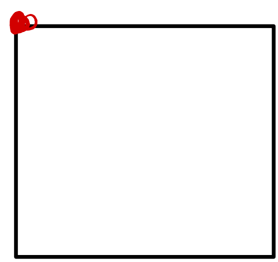
X 

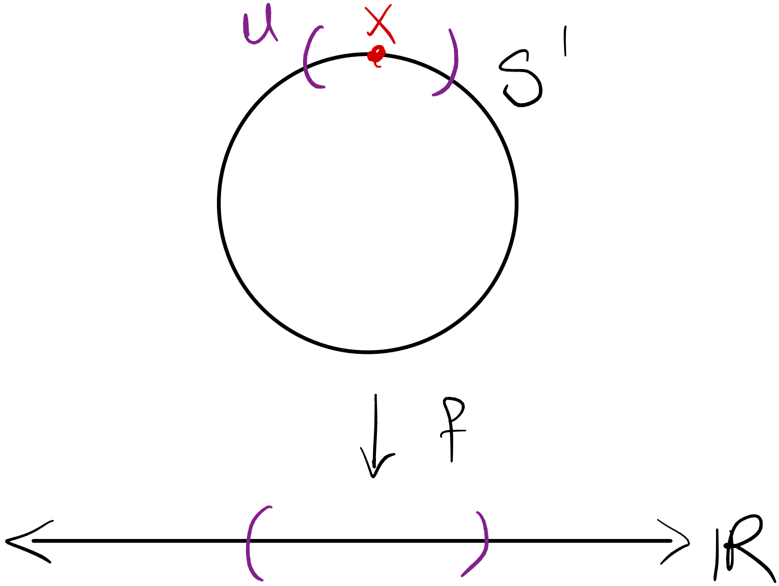
$$S^0 = \partial D^1 = \partial\left(\underset{x_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{x_2}{\circ}\right) = \{x_1, x_2\}$$

X 

② 1-dim. man.s:

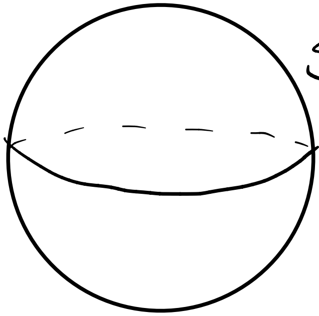
 $\mathbb{R}$

 $S^1 \cong$ 



U
x
S'
f
IR

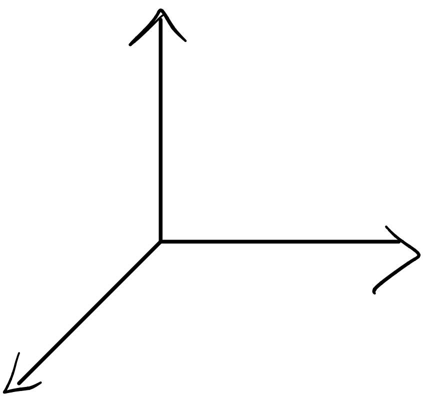
③ 2-dim. man. (surfaces)



S^2

$\mathbb{R}^2$

④ 3-dim. man.s

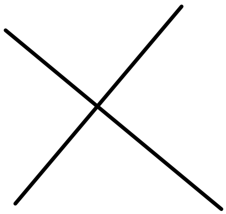


$\mathbb{R}^3$

S^3

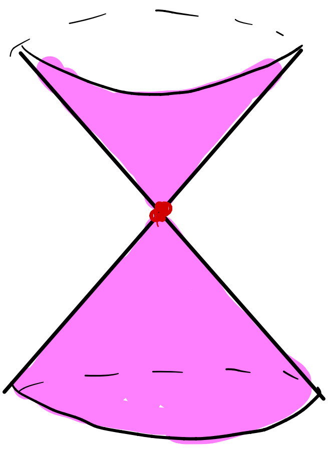
$S^1 \times S^2$

5



not a man.

(not even a topological space)



cone

not a manifold

(not even a topological space)

Q: How can we build "interesting" examples of 3-man., 4-man.

Product Manifolds

M, N : topological spaces.

$M \times N = \{ (x,y) : x \in M , y \in N \}$

Examples:

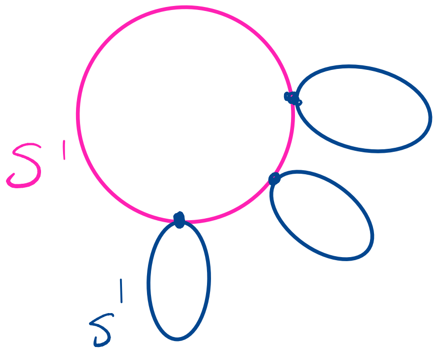
① $\mathbb{R}^2 = \underset{x}{\mathbb{R}^1} \times \underset{y}{\mathbb{R}^1}$ $(x,y) \in \mathbb{R}^2$

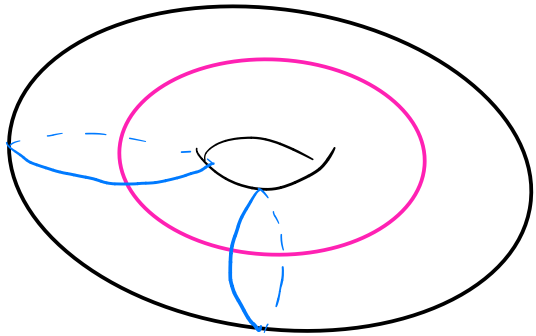
$\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^1$

$\vdots$

$\mathbb{R}^n$

② $S^1 \times S^1 = T^2$





Prop:

If M^m and N^n are manifolds,

$\Rightarrow M \times N$ is a $(m+n)$-dim. man.

3

$$T^2 \times S^1 = S^1 \times S^1 \times S^1 \qquad 3\text{-man.}$$

$$\underbrace{S^1 \times \cdots \times S^1}_{n\text{-many}} = T^n \qquad n\text{-man.}$$

Fact: $S^1 \times S^1 \times S^1 \underset{\text{homeo}}{\not\cong} \mathbb{R}^3$

4 $S^1 \times S^2$

Fact: $S^1 \times S^2 \underset{\text{homeo}}{\not\cong} \mathbb{R}^3$

5 $S^2 \times S^2$ 4-man.

6 $S^1 \times S^3$ 4-man.

Fact: $S^2 \times S^2 \underset{\text{homeo}}{\not\cong} \mathbb{R}^4$

$$S^1 \times S^3 \underset{\text{homeo}}{\not\cong} \mathbb{R}^4$$

$$\underset{I}{S^1} \longrightarrow X$$

$\pi_1(S^2) = 1$

$\{[\alpha]\}$

Q: Can we classify man. ?

Generalized Poincaré Conj:

$$X^n \underset{h.e.}{\sim} S^n \Longrightarrow X^n \underset{homeo}{\approx} S^n$$

$\pi_1(X) = 1$

$H_n(X) = 0 \quad \forall n \geq 2$

Smooth Poincaré Conj

$$X^n \underset{homeo.}{\approx} S^n \Longrightarrow X^n \underset{diffeo}{\cong} S^n$$

n	1	2	3	4	≥ 5
G.P.C.	easy	easy	Perelman	Freedman	Smale h-cobordism
S.P.C	easy	easy	easy	???	known depends on dim.
exotic = homeo. but not diffeo.					

X^n: closed, connected, oriented

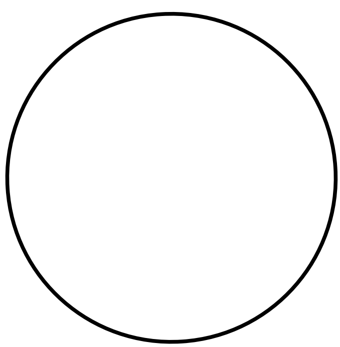
n-dim. manifold.

closed := compact and $\partial X = \emptyset$

Up to homeomorphism

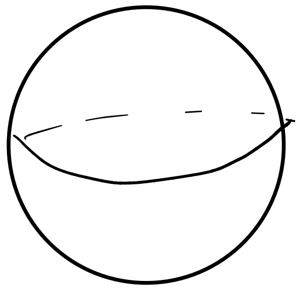
$n=1$: $\exists!$ topological man.

$$S^1 = \{ \vec{x} \in \mathbb{R}^2 : |\vec{x}| = 1 \}$$

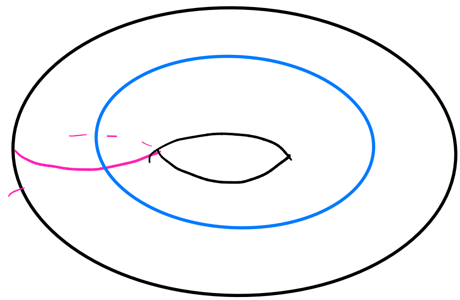


$n=2$:

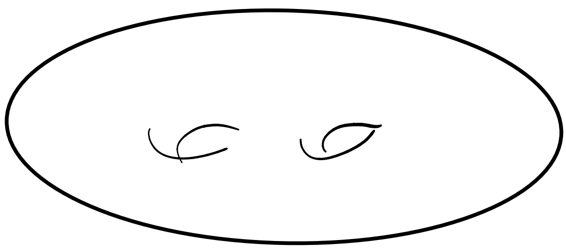
$S^2 = \Sigma_0$



$S^1 \times S^1 = T^2 = \Sigma_1$

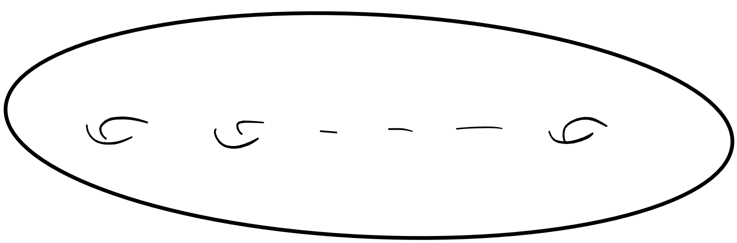


Σ_2



$\vdots$

Σ_g



$n=3$:

Thm [Moishezon]

Every top. 3-man admits

a unique smooth str.

Poincaré Conj. [Parelman]

M^3: closed, simply-conn. $\Longrightarrow M \underset{\text{homeo}}{\approx} S^3$

$n = 4$:

There are many simply conn. closed 4-man. that admit inf.ly many distinct smooth str.

There is NO know 4-man. which admit a unique smooth str.

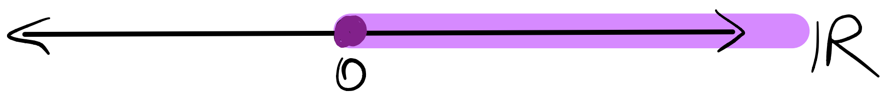
How about if $\partial X \neq \emptyset$?

Manifolds with boundary:

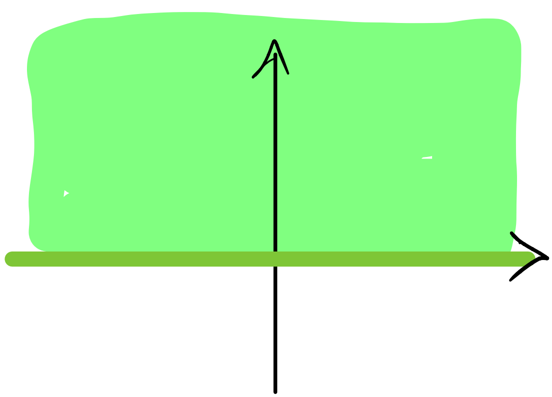
Defn: An n-dim. man. with boundary is a top. space which locally looks like upper half-space in $\mathbb{R}^n$.

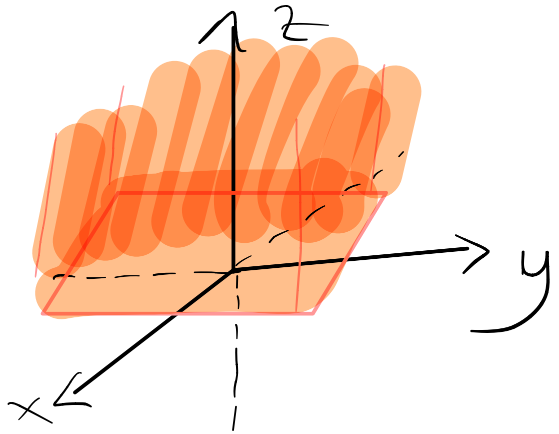
$$H^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

H^1_+



H^2_+





$\mathbb{R}^3$

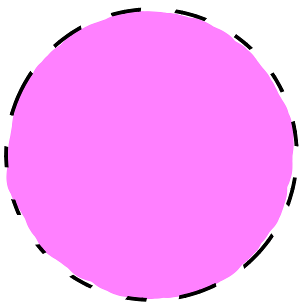
Defn:

The boundary ∂M of a man. M is the set of points which doesn't have any nbhd homeo. $\mathbb{R}^n$.

Examples:

① $D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

$\partial D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cong S^1$

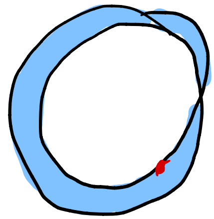


② $I = [a, b]$

$\partial(I) = \{a\} \cup \{b\}$

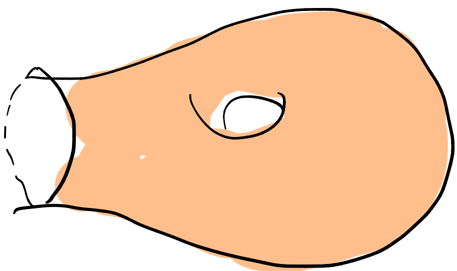
3 Möbius band

M



$\partial(M) \cong S^1$

4 Torus with one bdry comp.



M

5 Unit Ball

$$I^n = \underbrace{I \times I \times \cdots \times I}_{n\text{-times}} = B^n$$

$$I^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\} = B^n$$

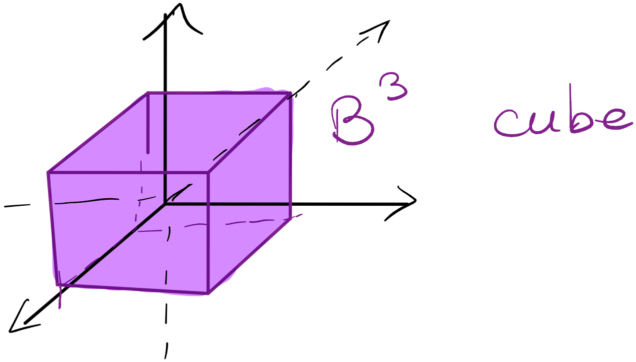
$$B^1 = \{x \in \mathbb{R} : x^2 \leq 1\} = [-1, 1] \cong [0, 1]$$

$$\partial B^1 = \{-1\} \cup \{1\} \cong S^0$$

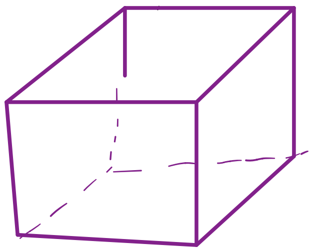
$$B^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} = D^2$$

$$\partial B^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} = S^1$$

$$B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 ; \ x_1^2 + x_2^2 + x_3^2 \leq 1\}$$



$$\partial B^3 =$$



$$\cong S^2$$

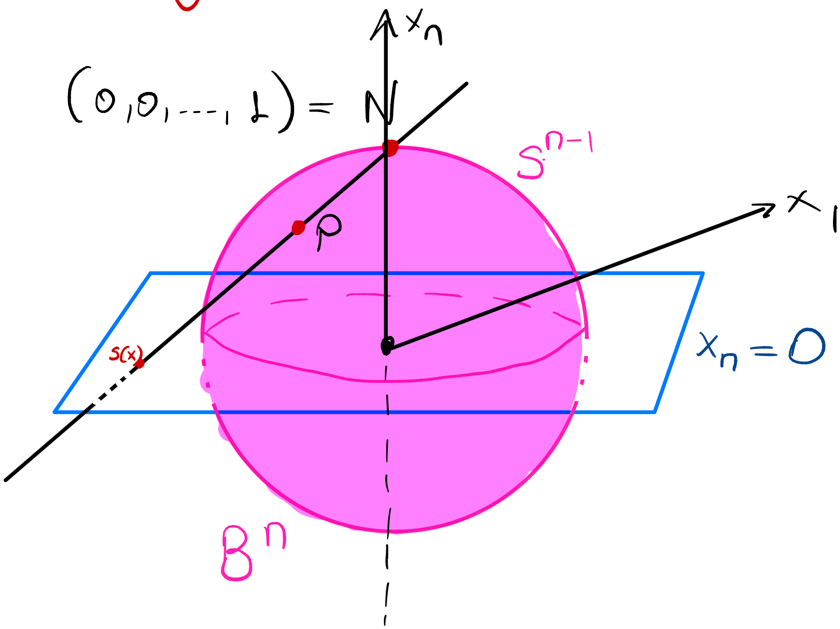
$$\partial B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 ; \ x_1^2 + x_2^2 + x_3^2 = 1\}$$

In general,

$$\partial B^n \cong S^{n-1}$$

$$\partial B^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1\} \cong S^{n-1}$$

Stereographic Projection:



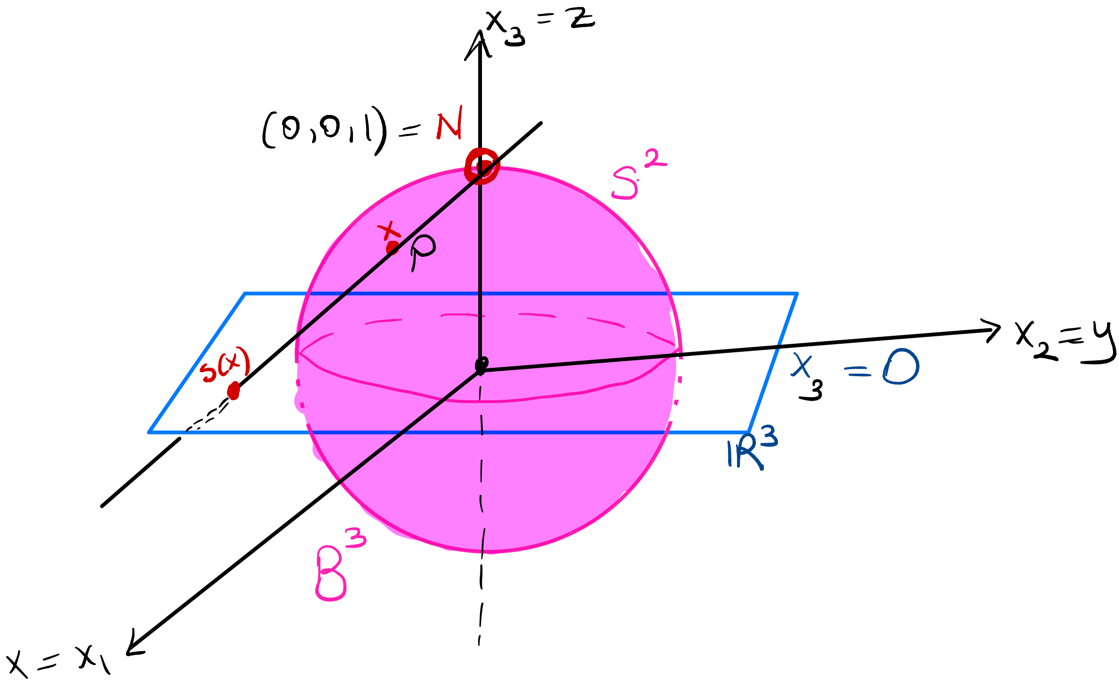
S^{n-1} intersects $x_n = 0$ along an equator

Connect N to P by a line.

That line int. $\{x_n = 0\}$ in exactly one point say $s(p)$

$s : S^{n-1} - \{N\} \longrightarrow \mathbb{R}^{n-1}$ homeomorphism.

$\Longrightarrow \mathbb{R}^n \cup \{pt\} \simeq S^n$ one point compactification



$$S : S^2 - \{N\} \longrightarrow \mathbb{R}^2 \qquad \text{homeomorphism.}$$

Submanifolds:

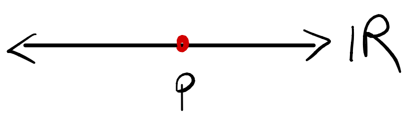
Let X^n be an n-man.

$Y^m \subset X^n$ is a submanifold if

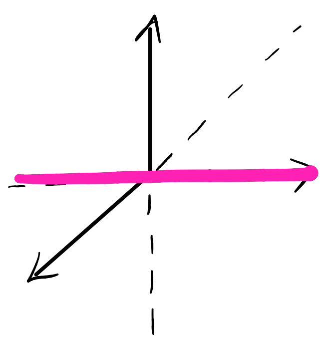
$\forall p \in Y$ $\exists$ nbhds $V_p \subset X$ and $U_p \subset Y$ s.t.

- $U_p \subset V_p$
- $(U_p, V_p) \underset{\text{homeo}}{\cong} (\mathbb{R}^m, \mathbb{R}^n)$

EX

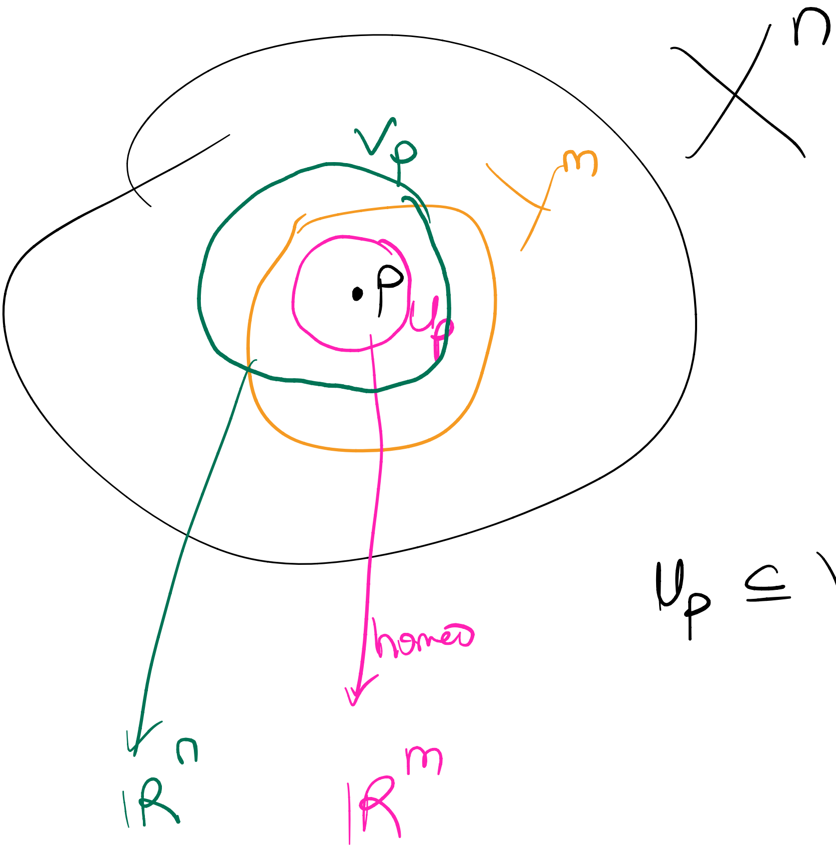


$Y^0 = \{p\}$ $\quad X^1 = \mathbb{R}^1$



$Y^1 = \{(x,y,z) \in \mathbb{R}^3 : y=0, z=0\}$

$X^3 = \mathbb{R}^3$

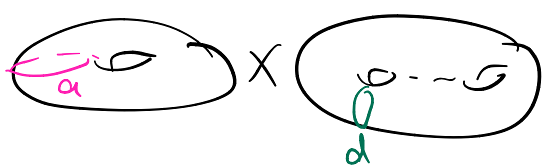


$$U_p \subseteq V_p$$

Ex: $S^1 \subset T^2 \times T^2$

$T^2 \times \{pt\}$

Ex: $T^2 \times \Sigma_g$



$$\underbrace{a \times d}_{T^2 = S^1 \times S^1} \subset T^2 \times \Sigma_g$$

Defn.: "surface"

A surface is defined to be

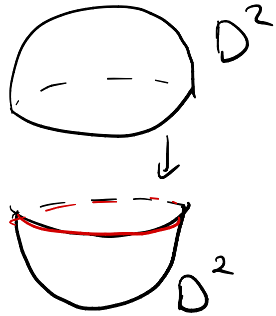
- a Hausdorff space
- $\forall x \in X \ \exists U \in \mathcal{N}(x)$ s.t. $U \simeq D^2 \subset \mathbb{R}^2$

Examples

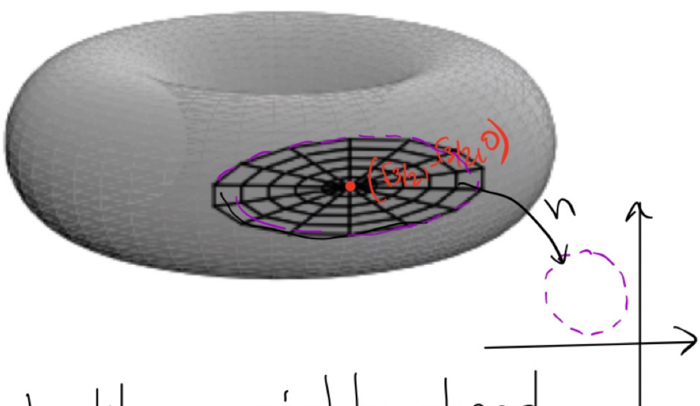
1. $\mathbb{R}^2$ is a surface

Since every pt. $(x,y) \in \mathbb{R}^2$ is contained in a neighbourhood say $D^2(x,y)$ which is an open disk in $\mathbb{R}^2$.

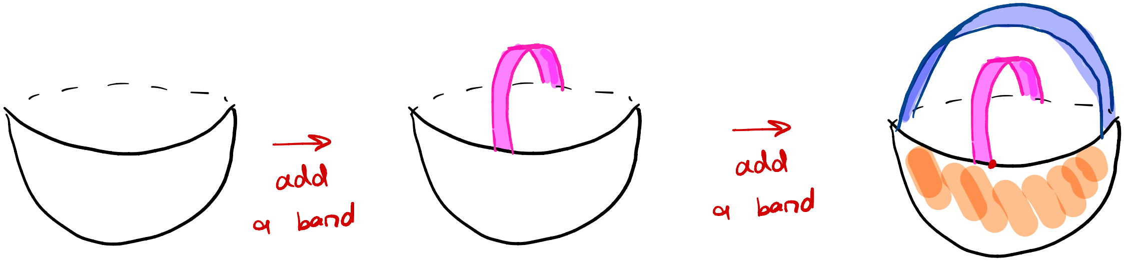
2 S^2

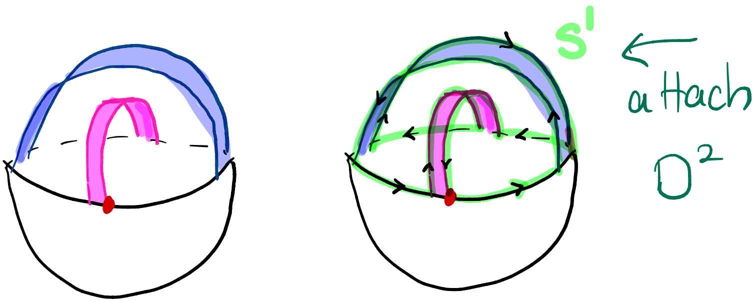


3 T^2 torus

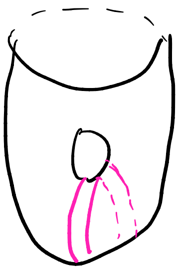


A disk-like neighbourhood about the pt. $(\sqrt{3}/2, -\sqrt{3}/2, 0)$.

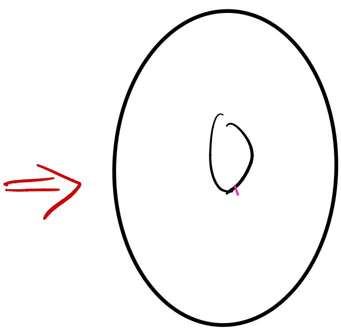




$\|$



cap it of with D^2



T^2

Connected Sum:

$$X_1^n \# X_2^n = [X_1 - D^n] \bigcup_{\varphi} [X_2 - D^n]$$

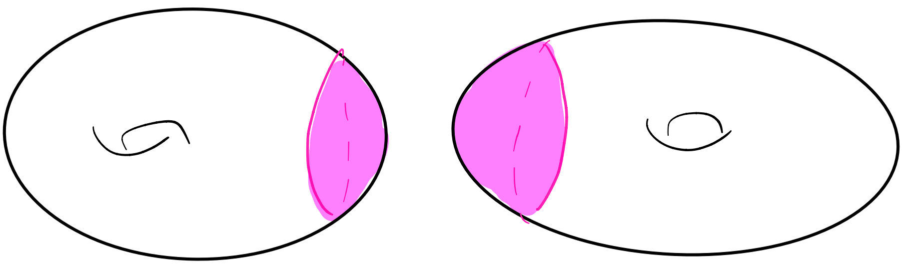
$\uparrow$ n-disk

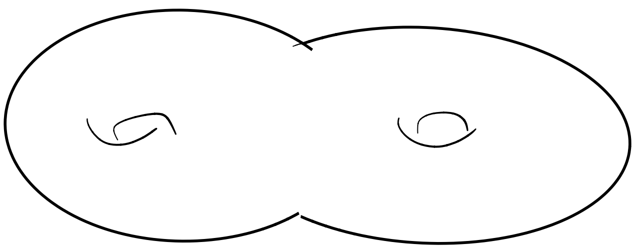
$$\partial D^n = S^{n-1}$$

$$S_1 \# S_2 = [X_1 - D^2] \bigcup_{\varphi} [X_2 - D^2]$$

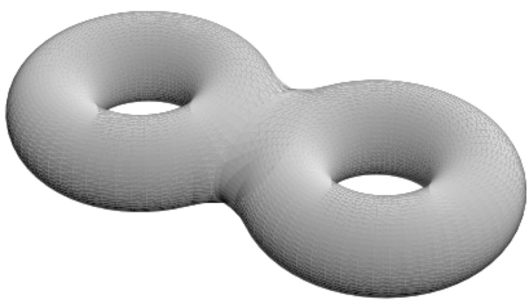
④ Σ_2

$T^2 \# T^2$

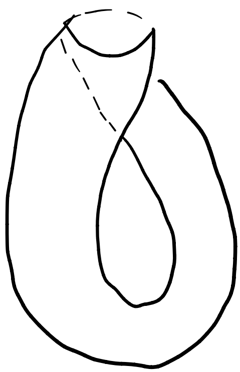




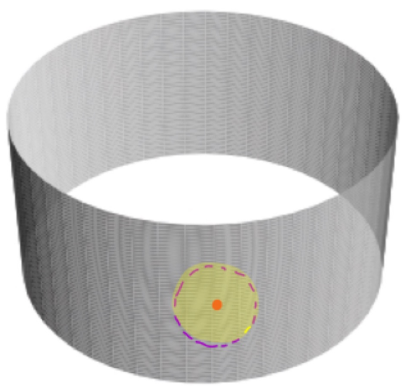
Σ_2



(5) Klein Bottle

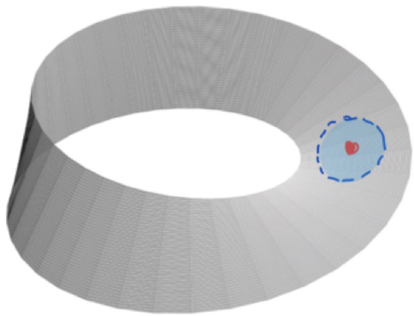


(6) Cylinder : $S^1 \times (0,1)$



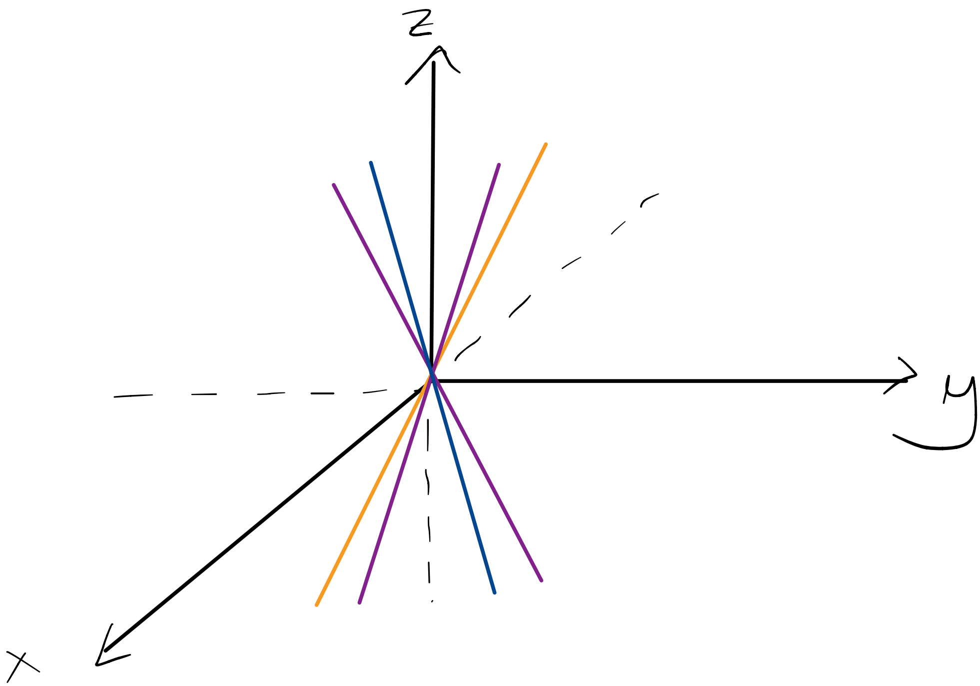
7 Möbius Band

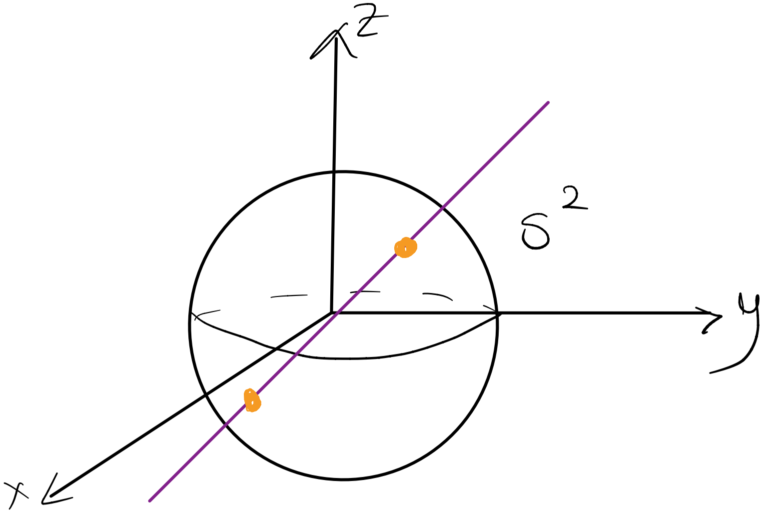
which is a surface obtained out of $S^1 \times (0,1)$ by cutting, twisting and regluing.



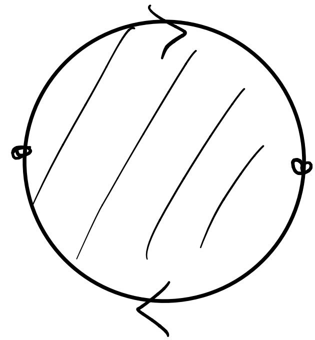
8) $\mathbb{RP}^2$: Real Projective Plane:

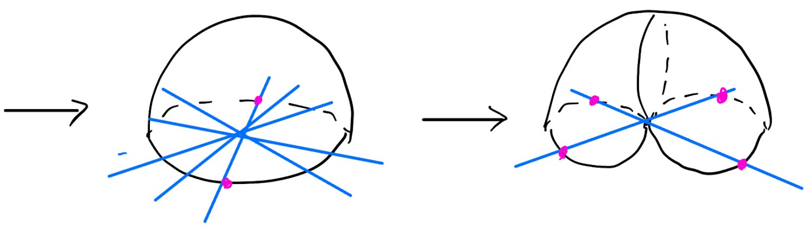
$$\mathbb{RP}^2 = \left\{ \begin{array}{l} \text{space of lines in } \mathbb{R}^3 \\ \text{through the origin.} \end{array} \right\}$$

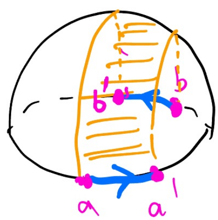




$$= S^2 \Big/ (x,y,z) \sim (-x,-y,-z)$$





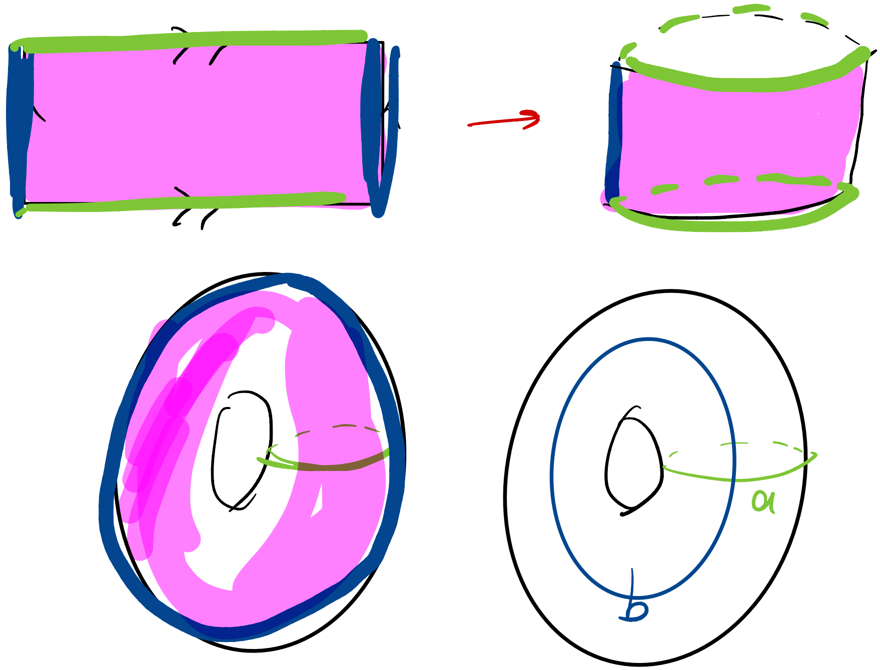


$a \sim b$
$a' \sim b'$

$MB \subset \mathbb{RP}^2$

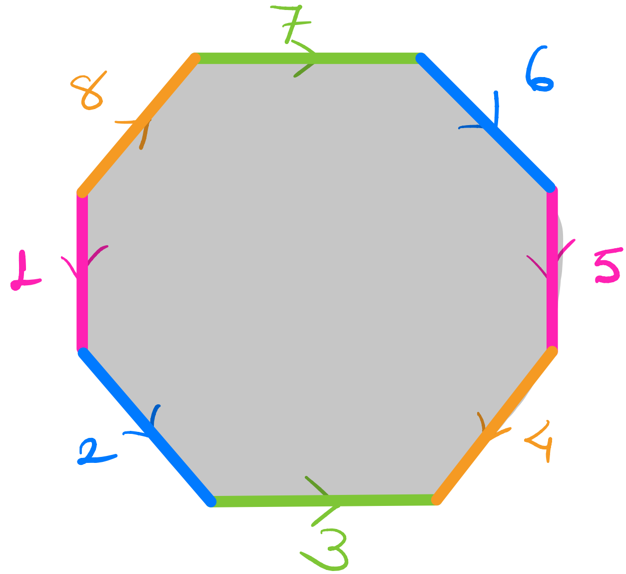
Q: What are the following surfaces?

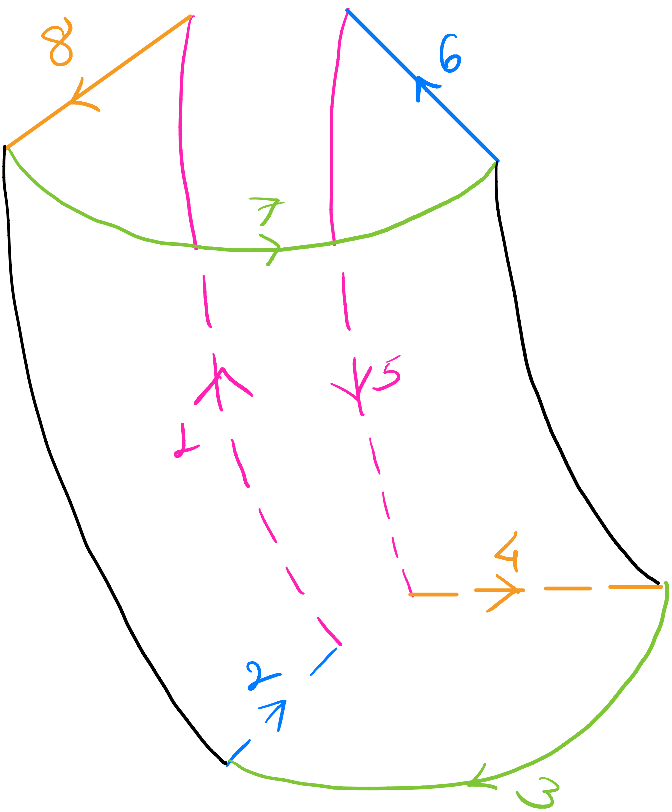
(1)

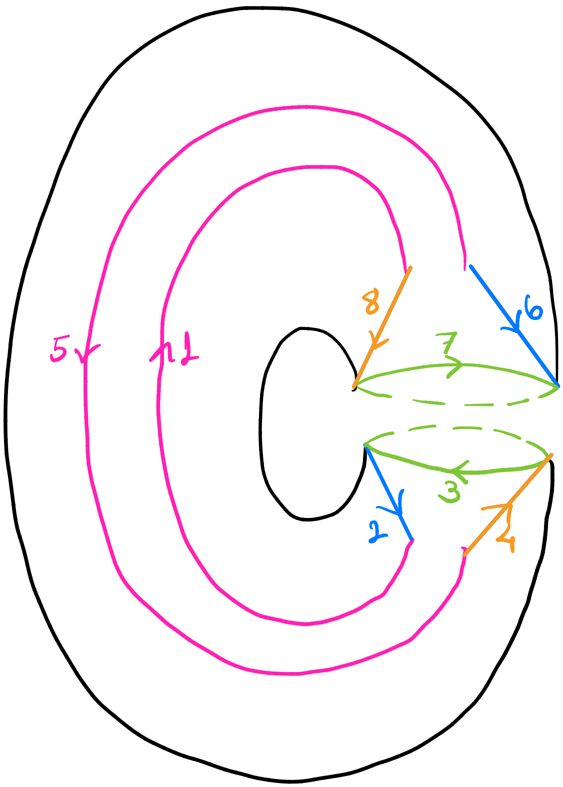


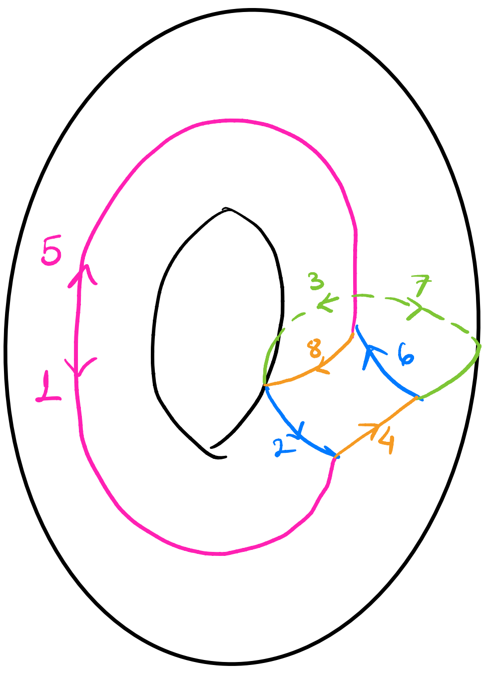
$H_1(T^2) = \langle a, b \,:\, \text{- -} \qquad = \mathbb{Z} \times \mathbb{Z}$

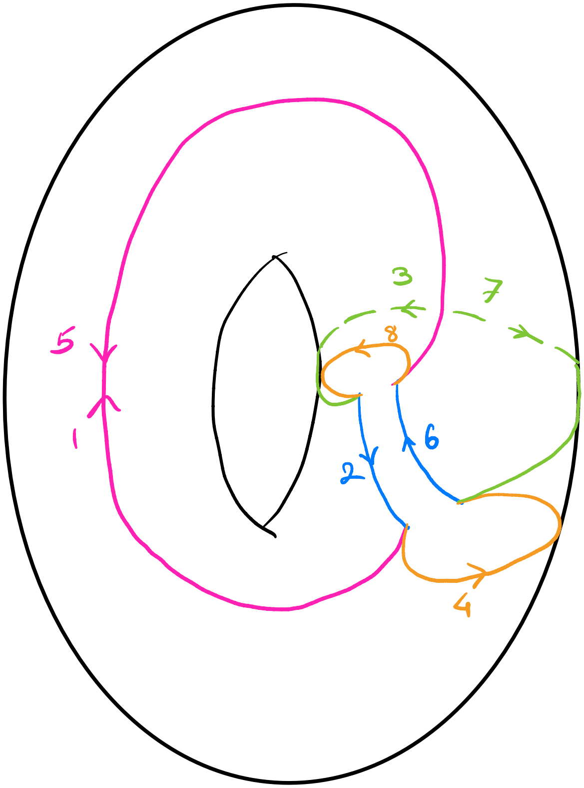
2

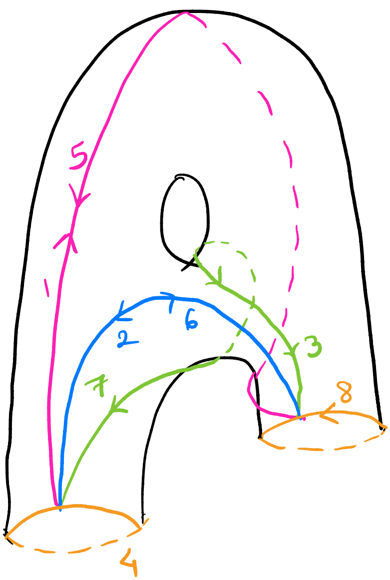


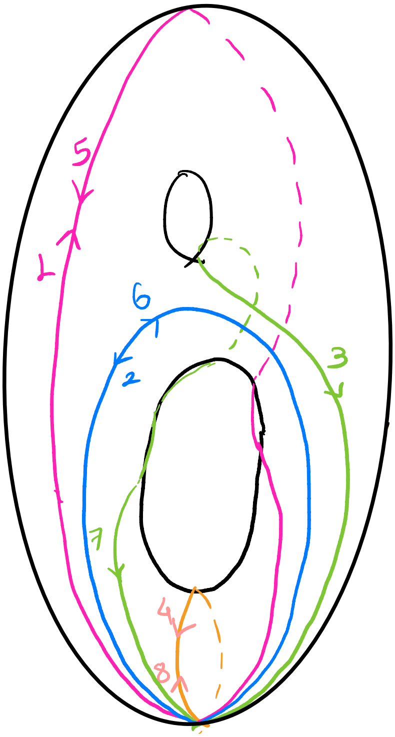


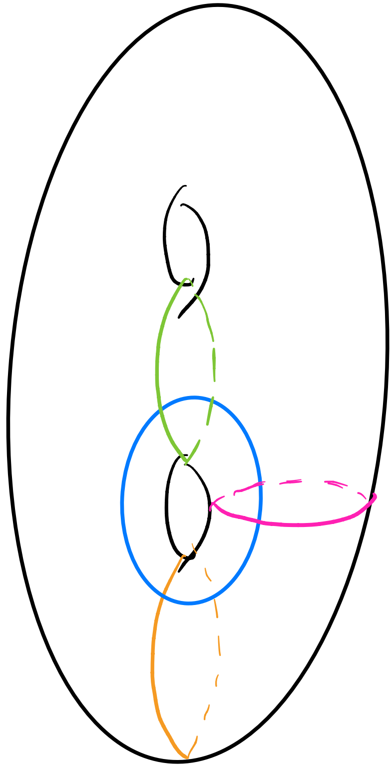

5
1
8
6
7
3
2
4


5
1
3
7
8
6
2
4

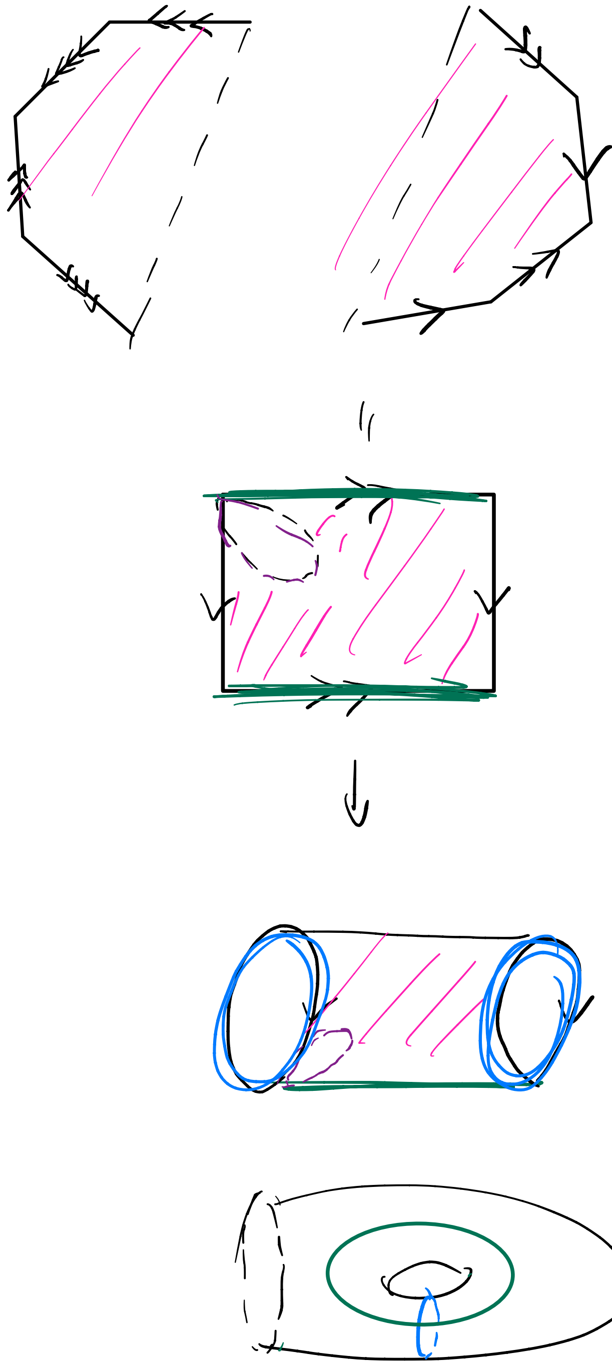

3
7
5
8
1
6
2
4


5
1
6
2
3
7
8
4

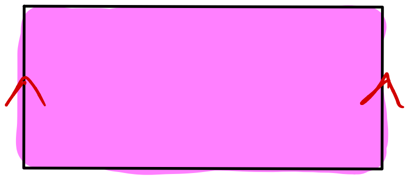

5
1
6
2
3
7
4
8



Another Way:

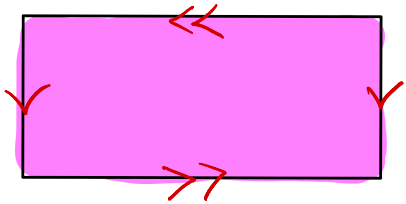


3



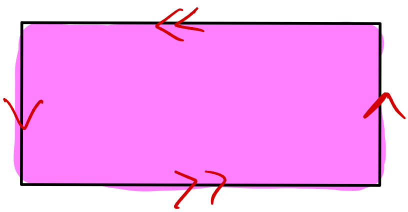
Cylinder

4



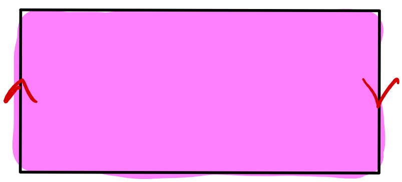
KB $\chi = 0$

5



$\mathbb{RP}^2$ $\chi = 1$

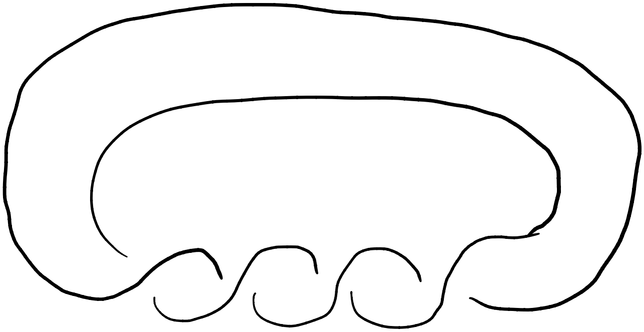
6



MB $\chi = 0$

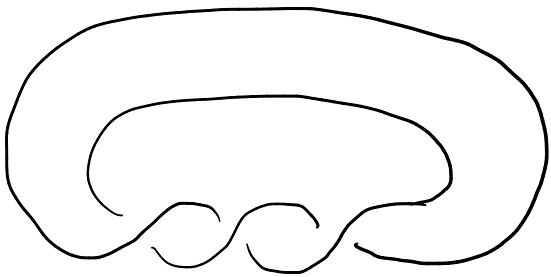
Some of these are NOT embedded submanifolds of $\mathbb{R}^3$.

7



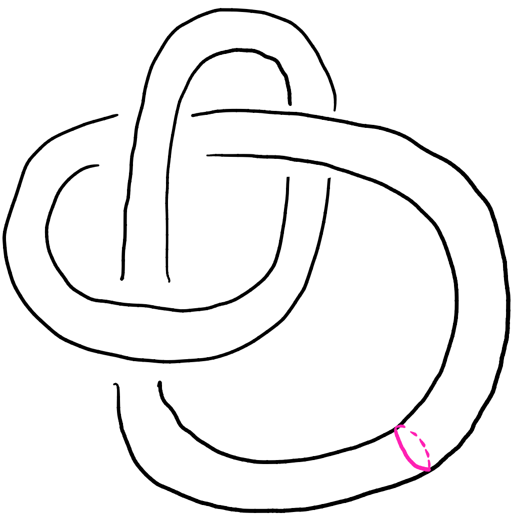
$\chi = 0$

8



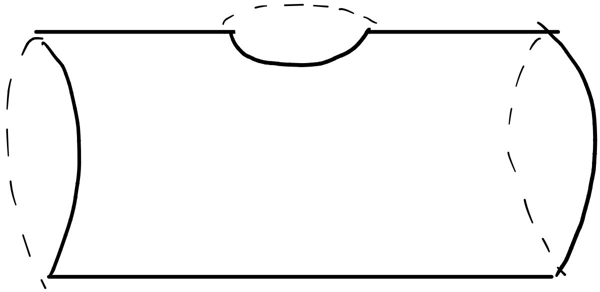
$\chi = 0$

9



$\chi = 0$

10



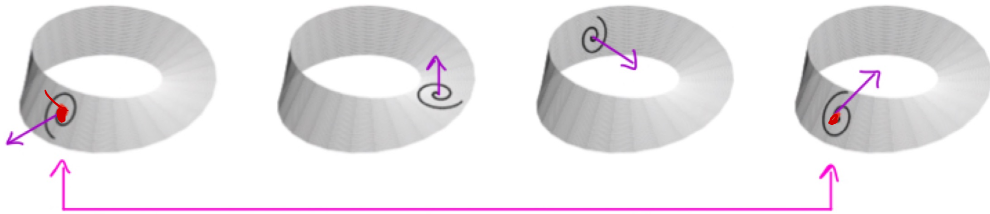
$$\chi = 1$$

Orientability

Slide a piece of paper along the surface and rotate it around.
If the nomal vector of the paper has the same direction when we arrive back at the same pt.,
no matter how we move around the surface,
then the surface is orientable.
Otherwise it is non-orientable.

orientable = "2-sided"
"has a back and front"

Example: Möbius band is non-orientable.



Ex: $\mathbb{RP}^2$ is NOT orientable.

Ex: $\mathbb{RP}^{2n}$ " " "

Ex: $\mathbb{RP}^{2n+1}$ is orientable.

Ex: Cylinder, S^2, T^2, Σ_2, ...

are all orientable.

Euler Characteristic

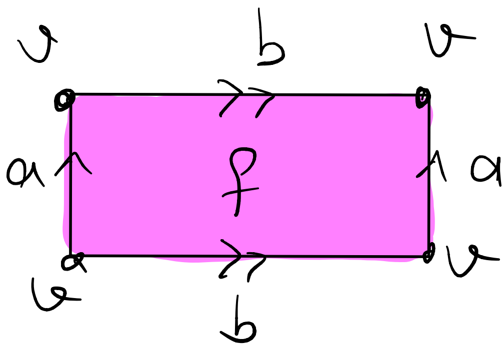
How to distinguish surfaces?

An integer "invariant" of man.s.

$$\boxed{\begin{aligned} &\chi(S) = V - E + F \\ &V := \#\text{ vertices} \\ &E := \#\text{ edges} \\ &F := \#\text{ faces} \end{aligned}}$$

Examples:

1



T^2

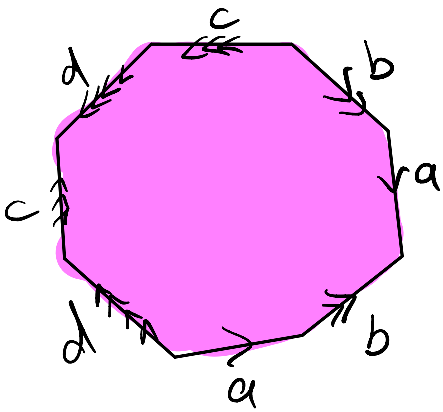
vertices: $\{v\}$

edges : $\{a, b\}$

faces : $\{f\}$

$\Rightarrow \chi(T^2) = 1 - 2 + 1 = 0$

2



Σ_2 8-gon

Σ_g 4g-gon

\# vertices = 1

\# edge = 4 $\longrightarrow$ 2g

\# faces = 1

$$\Rightarrow \chi(\Sigma_2) = 1 - 4 + 1 = -2$$

Q: Euler characteristic of $\underbrace{\text{a closed orientable surface}}_{\Sigma_g}$?

$\#$ vertices $= 1$

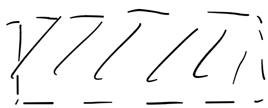
$\#$ edges $= 2g$

$\#$ faces $= 1$

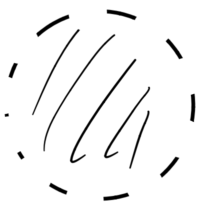
$$\Rightarrow \chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$$

$$\boxed{\text{surface} = \left(\bigcup \text{disks}\right) \cup \left(\bigcup \text{bands}\right)}$$

- vertice $\rightarrow$ disk (shrink)
- edge $\rightarrow$ band = an elongated disk

 $\simeq D^2$

- face $\rightarrow$ disk



\# vertices = 0

\# edges = 0

\# faces = 1

$$\chi(D^2) = 1$$

- boundary components

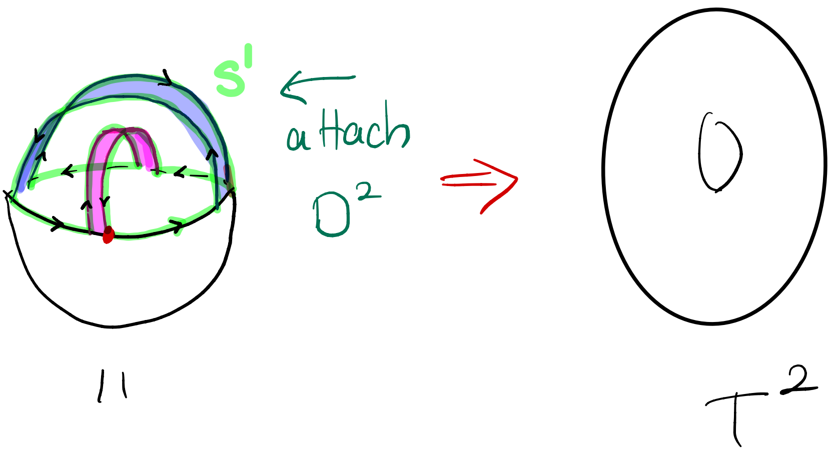
For connected, orientable surfaces

$$\boxed{\chi(S) = \#\underbrace{\text{disks}}_{\text{vertices and faces}} - \#\underbrace{\text{bands}}_{\text{edges}}}$$

Attaching a disk $\Rightarrow \chi + 1$

Attaching a band $\Rightarrow \chi - 1$

Example:



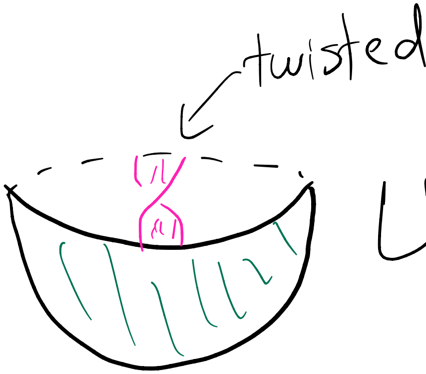
$\# \text{disks} = 2$

$\# \text{bands} = 2$

$\# \text{bdry comp.} = 0$

$$\Rightarrow \chi(T^2) = 0 - 2 + 2 = 0$$

Ex: (Class Exercise)

$P =$ 

$\cup \; D^2$

$$\chi(P) = 1 - 1 + 1 = 1$$

$$\boxed{\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2}$$

Example:

$$\chi(\Sigma_2) = \chi(T^2 \# T^2)$$

$$= \chi(T^2) + \chi(T^2) - 2$$

$$= 0 + 0 - 2 = -2$$

$$\chi(\Sigma_g) = 2 - 2g \underset{g=2}{=} -2$$

In general:

$$\chi(\Sigma_g) = \chi\left(\#_g T^2\right)$$

$$= g\,\chi(T^2) - (g-1)\cdot 2$$

$$= g \cdot 0 - 2(g-1) = 2 - 2g$$

Thm: A $\underbrace{\text{closed}}_{\text{compact} + \partial X = \emptyset}$, connected surface

is homeo. to exactly on of the following

non-orient: P, $\#_n P = P \# P \# \cdots \# P$

orientable: S^2, T^2, $\#_n T^2 = \Sigma_n$

Q: How do we distinguish these?

Q: Is Euler characteristic enough?

A: NO: we need orientability.

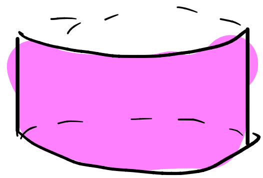
Thm: A compact, connected surface is homeo. to exactly one of the following surfaces:

- $\#_n P - \left(\bigcup_{i=1}^{m} D_i\right)$ OR
- $\#_n T^2 - \left(\bigcup_{i=1}^{m} D_i\right)$ OR
- $S^2 - \left(\bigcup_{i=1}^{m} D_i\right)$

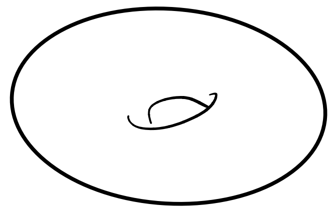
Q: How to distinguish these?

A: Euler characteristic is NOT enough even with orientability.

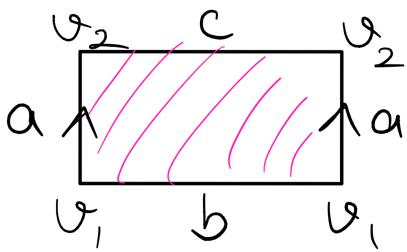
Ex:

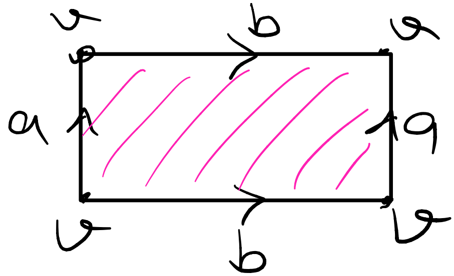


C



T^2





# vertices	2	1
# edges	3	2
# faces	1	1

$$\chi(C) = \chi(T^2) = 0$$

C, T^2: compact, orientable, connected

But $C \underset{\text{homeo}}{\not\cong} T^2$

$\partial(C) \neq \emptyset \qquad \partial(T^2) = \emptyset$